Decentralized Dynamic Matching with Signaling

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Abstract

We consider multi-period decentralized matching in a two-sided market of employers and employees (universities and students). Because of dynamics, asymmetry, and private information, we always observe delays and coordination failures with nonzero probability even in a simple case of two agents from each side. For this, we obtain conditions for different strategies to be in equilibrium. We show that problems of miscoordination and delay may be solved by signaling or by incentivizing immediate response that turn a dynamic problem into a static one and make the matching stable. Although implementing signaling or immediate response is always socially beneficial, it is not necessarily beneficial for the best university. In the general case, we obtain sufficient conditions for assortative matching to be and not to be in equilibrium.

Key words: *matching theory, dynamic matching, game theory, signaling, Nash equilibrium, stability*

1 Introduction

1.1 Overview

Many labor markets behave inefficiently because candidates (workers) are not ready to give full information about their preferences to potential employers (firms). In general, this may result in different coordination failures and delays. Employers remain unmatched, and some workers face a tough

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decision whether to accept an offer from a less preferred firm or wait for a better one.

By and large, the majority of the matching literature is devoted to a static approach, tackling the above-mentioned problem in a one-period setting. We introduce dynamics here, thus, reflecting a natural property of many markets that keep functioning after the initial matching has been made. Even in the simplest cases, dynamics significantly complicates the problem but also gives new insights and interesting results that would not exist in static matching settings.

In this paper, we consider multi-period decentralized matching with discounting, where remaining firms and workers who failed to match at the first period have a chance to match again. We start with the simplest case of two participants on each side of the market. Even this framework is quite sophisticated in terms of possible strategies. We find conditions on parameters of the model that lead to different equilibrium strategies for both firms and workers. It turns out that giving workers one more chance of matching in the second round (thus expanding their strategy sets) may surprisingly decrease their payoffs in some cases. The resulting problem may be solved by signaling before the first round. Signaling also helps to avoid delays and other coordination failures by turning a dynamic problem into a static one and making the matching stable and optimal for students. It is not necessarily true though in the case of more than two participants from each side. Also, students there may not be interested in truthful signaling.

Another way to tackle market delays and coordination failures is to incentivize students to report immediately within a round. We prove that encouraging students to immediately reject underwhelming offers also allows us to reach stability and optimality for them, plus it increases social welfare.

Since our model has a very natural application to Economics Job Market of PhD graduates (see, for example, Coles et al., 2010), we use it as a primary example. Throughout the paper, the employers' and the workers' sides of the market are called universities and students, respectively. Note that labor search and matching is not the only field of implementation, and the model may be applied to different markets. For example, in the case of dating sites, we can think about males and females, where one side makes offers and another side either accepts or rejects them. In this framework, "likes", "hearts", or "thumbs up" may be considered as signaling. Also, the same structure with some adjustments may be observed in markets of housing, child adoption, kidney exchange, and journal publications.

We start with defining the model in Section 2 and obtaining conditions on market participants' behavior in Sections 3 and 4. The "threat of reject" result of getting a lower payoff while expanding a strategy set is described in

Section 5. Section 6 is devoted to signaling, and in Section 7 we consider our matching model in a general case. Finally, immediate response is described in Section 8.

1.2 Related Literature

There are three different dimensions in the literature concerning matching, each of them consisting of two strands. The most obvious stratification is *static* and *dynamic* matching. Since our main interest is analyzing dynamic markets, we do not focus on the algorithms that may be applied to static problems (see Gale & Shapley, 1962, and Roth & Sotomayor, 1992). Nevertheless, at least one paper about one-period matching is worth mentioning: Coles, Kushnir, and Niederle (2013) propose a signaling mechanism similar to one described here. The design replicates the first round of the Economics Job Market. They prove that signaling fixes the problem of coordination failure, increases the number of matches and improves the welfare of workers. So does signaling in our dynamic model. Their model is also restricted to symmetric preferences, while our model allows for asymmetric ones.

The interest in dynamic matching has been significantly increasing for the last decade. While focusing on dynamic models, we can consider another dimension that reflects the level of regulation: *centralized* and *decentralized* markets. The vast majority of papers considers one of these poles: either a social planner is fully responsible for the entire matching procedure, or agents are not restricted at all while being completely self-interested. Although our model is fully decentralized, two papers examining centralized dynamic matching are worth mentioning. Akbarpour et al. (2020) investigate comparative characteristics of several algorithms in a very general market with continuous timing. They allow new agents to arrive and unmatched agents to perish. The authors prove that acting patiently in order to thicken the market and acting greedily in order to decrease a number of deaths is close to optimal for the social planner in the cases whether it knows if the agent is about to perish or it does not, respectively. Although the model is completely different, the idea of their Patient and Greedy algorithms is similar to patient and impatient students that we have in our model. In another recent work, Baccara et al. (2019) find an optimal mechanism for a market with two types of sequentially arriving agents. Thus, they allow some level of asymmetry that is similar to our model. Also, they consider partial decentralization leaving some (limited) decisions to market participants. According to the optimal mechanism, unpaired agents are put to the stock (up to a certain threshold) while all others are matched assortatively.

Finally, the last division concerns *Nash equilibrium approach* and *stabil-*

ity approach. Since in many decentralized matching models agents have no strategy sets and/or payoffs but only preferences order (due to the specifics of matching environment), stability is usually the only notion that allows to distinguish "bad" and "good" matchings. Narrowing down to dynamic markets, we should mention several papers relevant to ours. Doval (2020) explores a theory of stability with market opportunities changing over time (arriving agents). She introduces the notion of dynamic stability that captures the idea of trade-off between matching now and waiting for a better match in the second period. Ho (2019) considers dynamic two-sided many-to-one matching as a model that generalizes the college admission problem. He also introduces regret-free dynamic stability that always exists and, thus, is a more relaxed notion than Duval's dynamic stability. Liu (2018) studies repeated matching markets in a similar way, but the major difference with our model is that in his framework participants are fixed and they do not leave the market while matched. He shows that under some assumptions the problem of non-existence of stable matchings in a static environment disappears under repeated interactions. Probably, the closest framework to our paper is a publication by Pan (2018). She considers two-period decentralized matching with signaling and not fully revealed preferences but distinguishes exploding and regular offers (in our model all offers have the same timing) and focuses on market unraveling and stability. She proves that exploding offers generally lead to a less desirable social welfare and banning them is always profitable for high quality agents but not necessarily for low quality ones.

Table 1 reflects all three categories. In this paper, we consider decentralized dynamic matching, but the structure of this matching allows us to fruitfully use Nash Equilibrium along with stability notion.

Matching	**Centralized**	**Decentralized**
Static	Gale & Shapley (1962) Roth & Sotomayor (1992)	*Coles, Kushnir, Niederle (2013)*
Dynamic	*Akbarpour et al. (2019)* *Baccara, Lee, Yariv (2016)*	Doval (2018) Ho (2017) Liu (2019) Pan (2018)

Table 1: Papers that use (at least partially) Nash Equilibrium approach are in italics.

2 The Model

We start with describing the market with 2 participants on each side. The general case of n agents will be considered in Section 7.

Consider a two sided-market with two participants on each side. These participants form set $U = \{U_1, U_2\}$ of universities (or employers) and set $S = \{S_1, S_2\}$ of students (employees). The matching procedure lasts for an infinite number of periods with a discount factor $\delta \in [0, 1]$. In every period, universities can submit up to one offer. Students can either accept or reject offers; they cannot accept more than one offer during the whole process.

The valuation of students by universities is deterministic and common knowledge. Formally, define $u_k(l)$ as the utility university U_k receives by matching student S_l. Then

$$u_k(1) = w, \qquad u_k(2) = 1 - w \qquad (w \geqslant 1/2)$$

for any $k \in \{1, 2\}$. Thus, both universities have the same valuation of the students.

As for the students' side, let there be only two utility values, 1 and 2. Then, with probability p, each student values matching with university U_1 as 2 and with university U_2 as 1. Correspondingly, with probability $1 - p$, the student's utility of matching with U_1 (U_2) is 1 (2). Thus, with probability p each student independently likes the first university twice more than the second one, and does the opposite with probability $1-p$. Formally, we define $v_m(n)$ as the utility student S_m receives by matching the university U_n. Then

$$\begin{aligned} &Pr\{v_m(1) = 2\} = p, && Pr\{v_m(1) = 1\} = 1 - p, \\ &Pr\{v_m(2) = 2\} = 1 - p, && Pr\{v_m(2) = 1\} = p \end{aligned}$$

for any $m \in \{1, 2\}$.

Expected utilities for universities U_i and students S_j throughout the entire process are defined as $\mathbb{E}u_i$ and $\mathbb{E}v_j$, respectively ($i, j \in \{1, 2\}$).

Unmatched students and universities get a zero payoff. We summarize all the above-mentioned in Fig. 1.

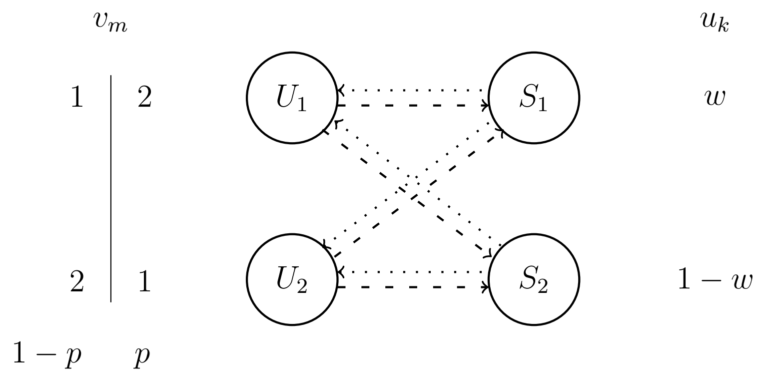

Figure 1: Universities can make an offer to any of the students (dashed arrows). Students may prefer any of the universities (with known probability p, dotted arrows), but can only react to universities' offers.

Since the distribution is asymmetric if $p \neq 1/2$, the first university is considered to be more prestigious on average. It does not mean though that students always prioritize it: one could subjectively prefer the second university because of location preferences, a cheaper housing rent, etc.

The timing in the model is the following.

0. Universities and students know their preferences. The preferences of universities are identical and certain to everyone. The distribution of students' preferences is common knowledge, but independent realizations of those random variables are private and accessible only to corresponding students.

1a. Universities choose to whom make an offer (see Fig. 2).

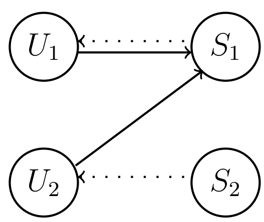

Figure 2: In this example, both universities decide to make an offer to the first (better) student. The first student likes U_1 more, and the second student prefers U_2.

1b. Students choose if they want to accept or reject offers from the universities (see Fig. 3).

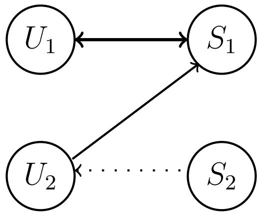

Figure 3: Bold double-headed arrow indicates successful matching.

1c. Matched pairs leave the market, and this is observable to remaining participants.

2. Remaining agents match at the second period with payoffs discounted by δ (see Fig. 4).

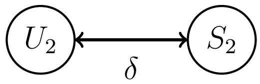

Figure 4: Matching in the second period when one pair has left the market after the first period.

3 (and beyond). The process continues until the market completely clears.

Now we describe the entire game formally.

Define the lattice $L = \{U_1U_2S_1S_2, U_1S_1, U_1S_2, U_2S_1, U_2S_2, \emptyset\}$ with a partial order $U_1U_2S_1S_2 \succ U_iS_j \succ \emptyset$ for any $i, j \in \{1, 2\}$. Elements of L represent the agents who are still active (unmatched). Let $\mathcal{L}$ be the set of all nonincreasing sequences starting from the maximum element:

$$\mathcal{L} = \{(l^n)_{n=1}^{\infty} : \forall n \in \mathbb{N} \ \ l^n \in L, l^n \succeq l^{n+1}, l^1 = U_1U_2S_1S_2\}.$$

Elements l^n describe the state of the game in period n. Then the strategy set of university i $(i = 1, 2)$ is $\mathcal{A}_i = \{A_i(l)\}_{l\in\mathcal{L}}$ — the set of sequences, where each element of a sequence is in its turn a simple set composed of two, one, or zero elements. Namely $(i, j \in \{1, 2\})$,

$$A_i(l) = \{A_i^1(l^1), A_i^2(l^2), \ldots, A_i^n(l^n), \ldots\}, \qquad l = (l^n)_{n=1}^{\infty},$$
$$\forall n \in \mathbb{N} \quad A_i^n(l^n) = \begin{cases} \{1, 2\} & \text{if } l^n = U_1U_2S_1S_2, \\ j & \text{if } l^n = U_iS_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, the elements of each set A_i^n are actions of U_i in period n: $a_i^n \in A_i^n$. For example, in the first period university i can make an offer to either student 1 $(a_i^1 = 1)$ or student 2 $(a_i^1 = 2)$.

We define the strategy set of student j ($j = 1, 2$) as $\mathcal{B}_j(l)$, where

$$\mathcal{B}_j(l) = (B_j^1(l^1), B_j^2(l^2), \ldots, B_j^n(l^n), \ldots),$$
$$\forall n \geqslant 0 \quad B_j^n(l^n) = \begin{cases} (\{1,2,0\}, \{1,0\}, \{2,0\}) & \text{if } l^n = U_1U_2S_1S_2, \\ \{i, 0\} & \text{if } l^n = U_iS_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

In the first case here, the first coordinate describes possible actions of the student if she gets two offers (accept U_1, U_2, or reject both). The second and the third coordinates describe the choice between accepting a single offer or rejecting it. In the second case, the student can get only one offer from the remaining university. Finally, in the third case the student has already matched and is not present on the market anymore.

Throughout the rest of the paper, by speaking about Nash equilibrium we consider a notion of sequential Nash equilibrium. It allows us to avoid non-credible threats and examine the game where agents do not commit to possible situations ex ante but make rational decisions in each step of the game (updating their beliefs if possible).

3 Students' behavior in equilibrium

Consider four options:

- A student gets two offers. Then she just accepts the preferable one (S_1 on Fig. 5).

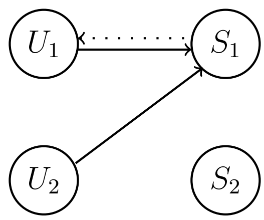

Figure 5: S_1 accepts an offer from U_1 as the preferable one and rejects an underwhelming offer from U_2. S_2 does not receive any offers and matches in the second round with U_2.

- A student gets zero offers. Then (S_2 on Fig. 5) she must wait for the second period where she matches the remaining university.

- A student gets exactly one offer, and this offer comes from her favorable university. Then she accepts the offer (Fig. 6).

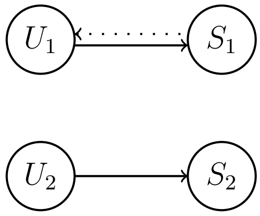

Figure 6: S_1 receives an offer from her preferable university U_1 and is happy to accept it.

- The most interesting case: A student gets exactly one offer, but from the less favorable university. Should she accept or reject it? (See Fig. 7 & 8.)

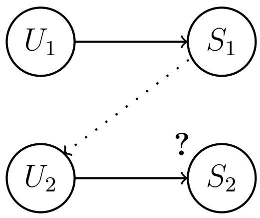

Figure 7: The first student gets an offer from U_1 although she prefers U_2. Since she does not receive any offer from the second university, she knows that this offer has gone to the second student. But the first student does not know what the preferences of the second student are.

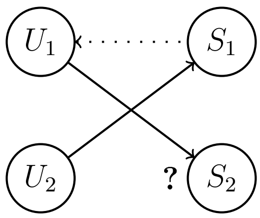

Figure 8: The same idea as in the previous figure, but here the first student gets an offer from the second university although she prefers the first one.

We want to understand what values of parameters are necessary and sufficient in equilibrium for a student to accept (reject) an offer in the last scenario. Assume the most interesting case when both students receive offers from the less favorable universities (see Fig. 9).

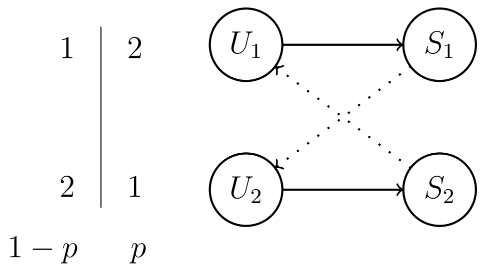

Figure 9: Both students get offers from the less favorable universities.

The next statement holds.

Proposition 1: *The subgame on Fig. 9 has only one equilibrium in the case of $\delta < \frac{1}{2-p}$: both students accept an unfavorable offer in the first period. In the case of $\delta \geqslant \frac{1}{2-p}$, there are three equilibria: both students accept in the first period, both students reject in the first period, and the mixed equilibrium where students reject with probabilities $\frac{1-\delta}{p\delta}$ and $\frac{1-\delta}{(1-p)\delta}$ correspondingly.*

Proof: Let x be the probability (belief) that a student who gets the only offer from the second university rejects it hoping to match the first university in the second period. Analogously, let y be the probability that a student who gets the only offer from the first university rejects it hoping to match the second university (that she prefers more) in the second period. Shortly,

$$x = Pr\{S_2 \text{ rejects } I_2 |\ S_2 \text{ prefers } I_1\}, \quad y = Pr\{S_1 \text{ rejects } I_1 |\ S_1 \text{ prefers } I_2\}.$$

S_1 chooses between accepting (and getting 1) and rejecting and getting $\delta((1-p)\cdot 1 + p\cdot(x\cdot 2 + (1-x)\cdot 1)) = \delta(1+px)$. S_2 in her turn chooses between 1 (accept) and $\delta(p\cdot 1 + (1-p)\cdot(y\cdot 2+(1-y)\cdot 1)) = \delta(1+(1-p)y)$ (reject).

Consider first the pure equilibria. Note that if one student accepts the worst offer ($x = 0$ or $y = 0$) then another student has to accept it as well (since she chooses between getting 1 and $\delta \leqslant 1$). Thus, the strategy of accepting in the first period is consistent for both students, and we have the first equilibrium $x = 0$, $y = 0$.

Let us assume now both students reject an offer in the first period ($x = 1$, $y = 1$). This strategy will be consistent if and only if $1 \leqslant \delta(1+p)$ and $1 \leqslant \delta(2-p)$. Since $p \geqslant 1/2$, the first inequality is redundant, and we thus have a necessary and sufficient condition $\delta \geqslant \frac{1}{2-p}$ for $x = 1$, $y = 1$ to be an equilibrium in this subgame.

The last step is to check if the subgame has any mixed equilibria. In this case, both students must be indifferent between two options, so we have a system of equations:

$$\begin{cases} 1 = \delta(1 + px), \\ 1 = \delta(1 + (1-p)y) \end{cases} \Rightarrow \begin{cases} x = \frac{1-\delta}{p\delta}, \\ y = \frac{1-\delta}{(1-p)\delta}. \end{cases}$$

To guarantee $x \leqslant 1$, $y \leqslant 1$, we again must have $\delta \geqslant \frac{1}{1+p}$ and $\delta \geqslant \frac{1}{2-p}$. Thus, the subgame has a mixed equilibrium if and only if $\delta \geqslant \frac{1}{2-p}$. ■

Corollary: Under $\delta \geqslant \frac{1}{2-p}$, the equilibrium where both students reject in the first period is the only Pareto optimal one.

Proof: It is easy to check that under $x = 0, y = 0$ and $x = \frac{1-\delta}{p\delta}$, $y = \frac{1-\delta}{(1-p)\delta}$ expected payoffs of the students are

$$\mathbb{E}v_1 = 1 + p, \qquad \mathbb{E}v_2 = 2 - p.$$

Under $x = 1, y = 1$, payoffs are

$$\mathbb{E}v_1 = 2p + \delta(1-p)(1+p) \geqslant 2p + \frac{1-p^2}{2-p} = -2 + p + \frac{5}{2-p} > 1 + p,$$
$$\mathbb{E}v_2 = 2(1-p) + \delta p(2-p) \geqslant 2(1-p) + p = 2 - p,$$

that proves the statement. ■

Note: The case on Fig. 8 may be considered the same way and delivers absolutely the same results.

Thus, we can say that under $\delta < \frac{1}{2-p}$ students tend to be impatient and accept any offer in the first period. On the contrary, under $\delta > \frac{1}{2-p}$ students tend to behave patiently hoping to get a better match in the second period (see the graphical interpretation on Fig. 10).

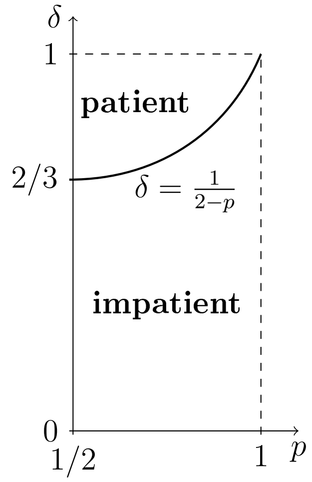

Figure 10: Students tend to be patient and wait for a better offer in the upper area with larger δ. On the contrary, in the lower area they accept any offer in the first period.

4 Universities' behavior in equilibrium

4.1 Patient students: $\delta > \frac{1}{2-p}$

Examine the strategies of universities in the case of $\delta > \frac{1}{2-p}$. Under these conditions, they know that students tend to reject an underwhelming offer in the first period (selecting the student-optimal equilibrium in the subgame). Each university chooses between strategies 1 and 2 of making an offer to S_1 and S_2, respectively. The universities' expected payoff matrix for the entire game along the equilibrium path is the following:

$U_1 \setminus U_2$	$a_2^1 = 1$	$a_2^1 = 2$
$a_1^1 = 1$	$\boldsymbol{pw + \delta(1-p)(1-w)}$ $\boldsymbol{(1-p)w + \delta p(1-w)}$	$pw + \delta(1-p)(p(1-w) + (1-p)w)$ $(1-p)(1-w) + \delta p((1-p)w + p(1-w))$
$a_1^1 = 2$	$p(1-w) + \delta(1-p)(pw + (1-p)(1-w))$ $(1-p)w + \delta p(pw + (1-p)(1-w))$	$p(1-w) + \delta(1-p)w$ $(1-p)(1-w) + \delta pw$

It can be proven that 1 is a dominant strategy for both universities in the first period. Thus, $a_i^1 = 1$ $(i \in \{1,2\})$ are the only possible actions along the equilibrium path: both universities tend to make an offer to the best student. For brevity, we further denote our equilibrium strategies for the universities in a vector form. Here it is equivalent to $(1,1)$.

Students' expected payoffs in this case:

$$\mathbb{E}v_1 = 2, \qquad \mathbb{E}v_2 = \delta(1 + 2p(1-p)).$$

4.2 Impatient students: $\delta < \frac{1}{2-p}$

Under these conditions, universities know that students accept an underwhelming offer in the first period. The universities' expected payoff matrix for the entire game along the equilibrium path is the following:

$U_1 \setminus U_2$	$a_2^1 = 1$	$a_2^1 = 2$
$a_1^1 = 1$	$pw + \delta(1-p)(1-w)$ $(1-p)w + \delta p(1-w)$	w $1-w$
$a_1^1 = 2$	$1-w$ w	$p(1-w) + \delta(1-p)w$ $(1-p)(1-w) + \delta pw$

Consider three cases:

(I)

$$\begin{cases} pw + \delta(1-p)(1-w) > 1-w, \\ (1-p)w + \delta p(1-w) > 1-w. \end{cases}$$

(II)

$$\begin{cases} pw + \delta(1-p)(1-w) > 1-w, \\ (1-p)w + \delta p(1-w) < 1-w. \end{cases}$$

(III)

$$\begin{cases} pw + \delta(1-p)(1-w) < 1-w, \\ (1-p)w + \delta p(1-w) < 1-w. \end{cases}$$

In case (I), $a^1 = 1$ is a dominant strategy in the first period for both universities. As in 4.1, $(1,1)$ is the only pair of actions along the equilibrium path here: both universities make an offer to the best student.

Students' payoffs:

$$\mathbb{E}v_1 = 2, \qquad \mathbb{E}v_2 = \delta(1 + 2p(1-p)).$$

In case (II), 1 is a dominant strategy in the first period for U_1; the rational second university plays 2. The only pair of actions along the equilibrium path here is $a_1^1 = 1, a_2^1 = 2$ or $(1,2)$: both universities match assortatively.

Students' payoffs:

$$\mathbb{E}v_1 = 1 + p, \qquad \mathbb{E}v_2 = 2 - p.$$

In case (III), there is no dominant strategy in the first period. We have three Nash equilibria, similar to the Hawk-Dove game: $(1,2)$ with $\mathbb{E}u_1 = w$, $\mathbb{E}u_2 = 1-w$, $(2,1)$ with $\mathbb{E}u_1 = 1-w$, $\mathbb{E}u_2 = w$, and a mixed one with $\alpha = Pr\{a_1^1 = 1\}$, $\beta = Pr\{a_2^1 = 1\}$, where

$$\alpha = \frac{w - (1-p)(1-w) - \delta pw}{(1-\delta)p}, \qquad \beta = \frac{w - p(1-w) - \delta(1-p)w}{(1-\delta)(1-p)}.$$

A Pareto optimal equilibrium does not exist, so it is very hard to predict the possible behavior of rational universities here. It is clear though that they try to avoid competition under any circumstances.

Thus, we come up with three scenarios in the first round:

(I) Competition. The second university tries to "steal" a better student from the first university. $(1,1)$ is the only equilibrium here.

(II) Assortative Matching. The second university yields and makes an offer to a worse student. The only equilibrium is $(1,2)$.

(III) Hawk-Dove. Three equilibria are possible: $(1,2)$, $(2,1)$, and a mixed one.

Depending on different w, these sets have a different structure. For example, note that (III) does not exist if $w \geqslant 2/3$ (universities value the best student twice or more than the worst one). It can be described graphically (see Fig. 11).

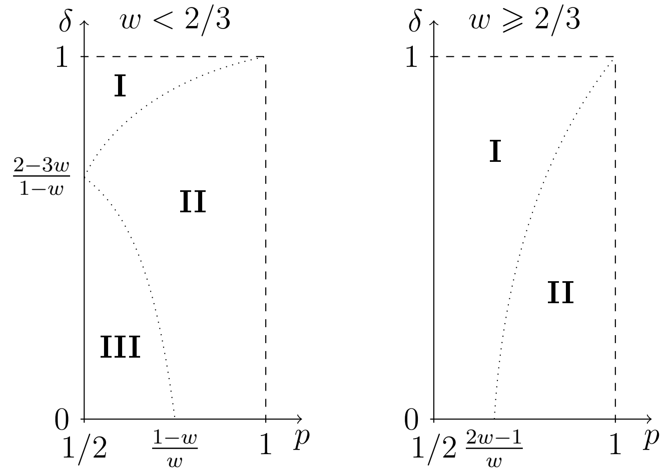

Figure 11: Areas of universities' behavior for different w and impatient students.

Finally, taking into account both students' and universities' behavior sets and overlapping them on the same graph, we get a full graphical representation of the matching process in equilibrium (Figg. 12 – 14).

$1/2 \leqslant w < 4/7$

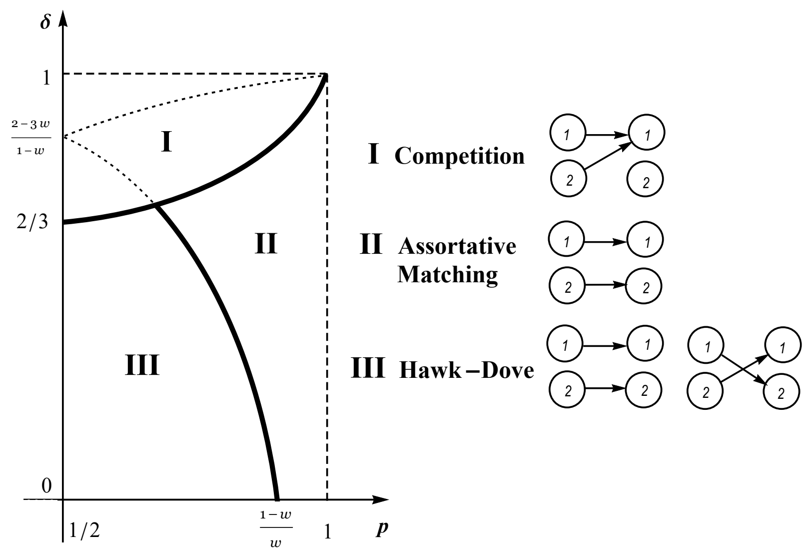

Figure 12: If students do not differ much, then the second university is interested in the best student only if discount factor δ is sufficiently high and probability of choosing the best university p is sufficiently small (area I). In all other cases, it prefers to match in the first period using the strategy opposite to the first university.

$4/7 \leqslant w < 2/3$

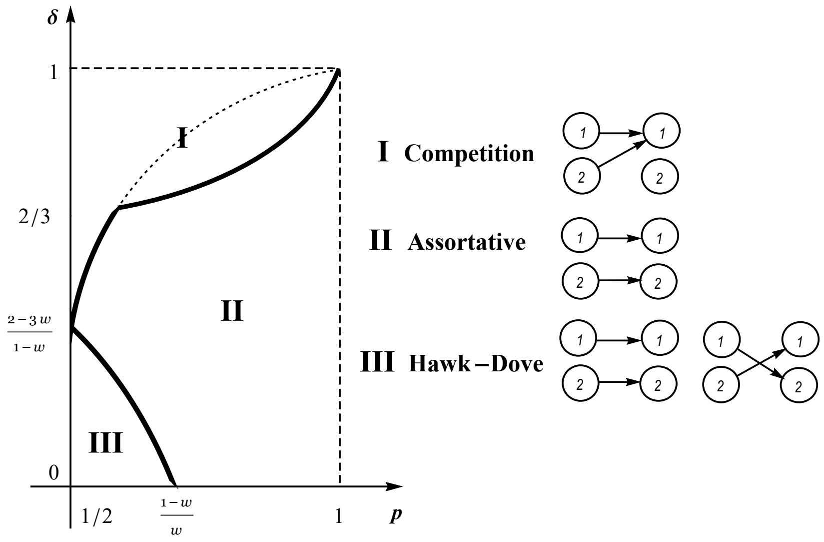

Figure 13: If students differ more significantly, "stealing" becomes more appealing under small values of p and sufficiently high values of δ.

$2/3 \leqslant w < 1$

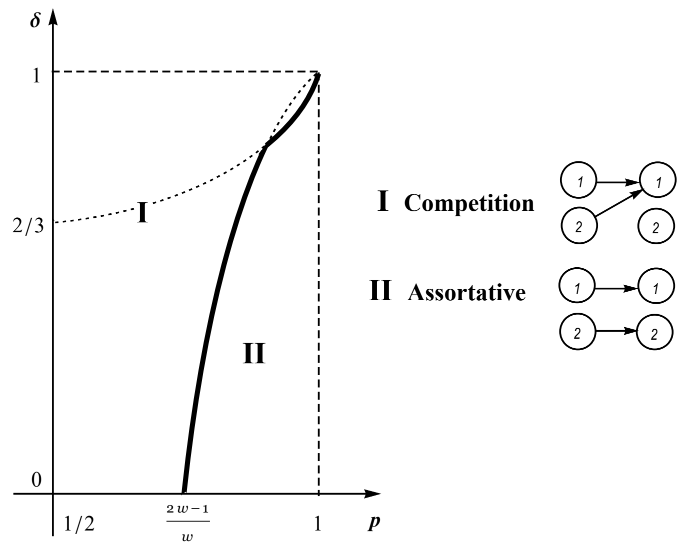

Figure 14: The difference in the level of students is huge (more than twice), so the second university tries to "steal" the best student under almost all values of parameters except very high p, where chances to be chosen by a better student are low.

Note that in case I we experience delay, because one of the pairs (containing the worst student) will necessarily form only in the second round. In cases II and III, we may experience coordination failure and unstability. In assortative matching, the first student always accepts an offer from the first university due to impatience. If she prefers the U_2, we face unstable matching: both the first student and the second university would like to quit existing matching and form a new one. Both these problems (delay and coordination failure) may be solved by implementing signaling (see section 6) or incentivized reports within a round (section 8).

5 Threat of Reject

In many games, while expanding a strategy set for one of the players, we can expect that this player's payoff will increase (or at least stay the same). The more options (and more flexibility) the player has, the better she performs. In the worst case scenario, she can just play the strategy from the original set and get the same utility. However, this expanded strategy set may affect the behavior of another player and, thus, lead to completely different equilibrium paths. Under some values of parameters, this is exactly the case in our matching process. We call this result "a threat of reject", because giving more options to some players may credibly threaten other participants and force them to choose different strategies in equilibrium that lead to worse payoffs for all parties involved.

We first introduce the idea of "threat of reject" by a simple example.

Example. Let $w = 5/9$, $\delta = 5/6$, and $p = 3/4$. In our model, the expected utility of U_2 if it makes an offer to S_2 in the first period is $\mathbb{E}u_2 = (1-p)(1-w) + \delta p((1-p)w + p(1-w)) = 0.406$. However, making an offer to the best student S_1, it receives an expected payoff $\mathbb{E}u_2 = (1-p)w + \delta p(1-w) = 0.417$. Since in the second case utility is higher, U_2 prefers to compete rather to try to match assortatively. In this case, the second student remains with no offer in the first round, and her expected utility is $\mathbb{E}v_2 = \delta(1 + 2p(1-p)) = 1.146$ (see Fig. 15).

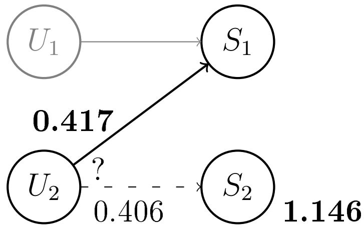

Figure 15: Expected payoffs of U_2 and S_2 in the regular case.

Now consider a situation, where reject is not allowed. Students are not given any option to choose, so they have to accept any single offer they get (but they still can reject an underwhelming offer in the case they get two). Then the expected utility of U_2 by making an offer to S_2 becomes higher: $\mathbb{E}u_2 = 1 - w = 0.444$. Making an offer to S_1 gets paid the same amount: $\mathbb{E}u_2 = 0.417$. In this case, the second university chooses not to compete, and the second student gets her first round expected payoff $\mathbb{E}v_2 = 2 - p = 1.25$ (see Fig. 16).

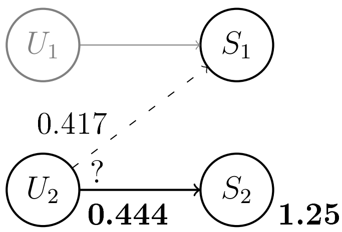

Figure 16: Expected payoffs of U_2 and S_2 in the case when reject is not allowed.

We can see that in the second case with prohibited reject both U_2 and S_2 receive higher payoffs. Thus, giving more options to the student does not necessarily help her to get a higher utility. ■

Let us prove this result in a general case.

Proposition 2 (Threat of Reject): *Consider the area*

$$\begin{cases} \delta > \dfrac{1}{2-p}, & (1)\\ (1-p)w + \delta p(1-w) < 1-w, & (2)\\ \delta \leqslant \dfrac{2-p}{1+2p-2p^2}. & (3) \end{cases}$$

Under values of parameters (1)–(3), allowing students to reject decreases the expected utility of the worse student.

Proof: (1) tells us that students are going to reject an underwhelming offer. (2) means that if they are not, the second university would be happy to choose the second student rather than to compete for the first one. Finally, (3) reflects the idea that the second student gets worse while waiting for the second round rather than matching in the first period: $\delta(1+2p(1-p)) < 2-p$. ■

The "threat of reject" area for small $1/2 \leqslant w < 4/7$ is represented on Fig. 17. For $w \geqslant 4/7$, the idea is the same.

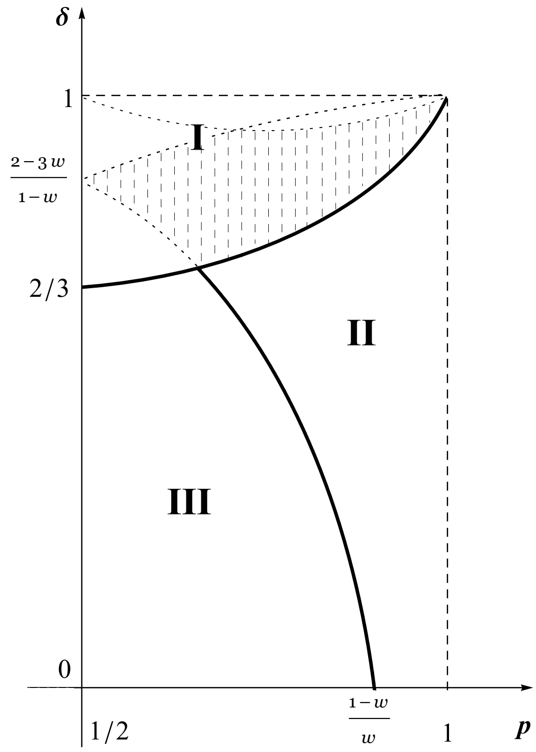

Figure 17: The "threat of reject" area (hitched) with point $\delta = 5/6$, $p = 3/4$ inside in the case of $1/2 \leqslant w < 4/7$.

The intuition behind this paradox is the following. Taking into account that the discount factor is sufficiently large *and facing the possible reject in the first period*, the second university finds it more appealing to try to "steal" a better student (otherwise it will still get the second one with only a slightly discounted payoff). The problem can be solved by introducing some commitment design or by signaling.

6 Signaling

Let us now introduce signaling from students to universities in a way similar to defined by Coles, Kushnir, Niederle (2013).

- Before offers take place, students can send up to one signal to any university.
- Signaling is private in the sense that other market participants (except the student herself and the aimed university) do not have any information about it.

Note that for both students truthful signaling is a weakly dominating strategy (over staying idle or sending a false signal). Thus, we assume for simplicity that both students always make truthful signals under any values of parameters.

Taking all the above-mentioned into account, we can see now that in equilibrium signaling turns a dynamic problem into a static one.

Proposition 3: *In the model with signaling, all the participants necessarily match in the first round in equilibrium. This matching is stable.*

Proof: The university that received a signal from the best student makes an offer to her. Thus, S_1 always gets what she wants. Knowing that, the university which did not receive a signal from S_1 does not have any incentive to make an offer to her. Hence, this university makes an offer to S_2, who in her turn immediately accepts the offer. That happens because S_2 may receive only up to one offer, and she understands that in the second round there will be no other offers.

This matching is stable in the sense that there is no blocking pair. The one containing the best student is completely satisfied with their choice: the student gets a preferable university, and the university gets the best student. ■

Note that not only the first student gets a higher payoff thanks to signaling ($v_1^s = 2$ — we use superindex s for utilities in the model with signaling), but the second student improves at least in cases I and II. Indeed, after signaling $\mathbb{E}v_2^s = 1+2p(1-p)$, which is higher or equal than $\mathbb{E}v_2 = \delta(1+2p(1-p))$ in case I and $\mathbb{E}v_2 = 2-p$ in case II without signaling. Thus, signaling eliminates the "threat of reject" problem, delivering all the students higher expected payoffs. In case III, however, things become intricate: for example, equilibrium $(2,1)$ can imply $\mathbb{E}v_2 = 1+p$ which is higher than $\mathbb{E}v_2^s$. Nevertheless, the total utility of students (students' social welfare) with signaling is always higher or equal than one without signaling.

Also note that since signaling necessarily makes the matching process stable, the first university may not be interested in implementing that. However, the total utility of both universities with signaling is equal or higher than the one with no signaling. Indeed, signaling implies $\mathbb{E}u_1^s = pw + (1-p)(1-w)$, $\mathbb{E}u_2^s = (1-p)w + p(1-w)$, and the total welfare of the universities is equal to 1. In case I, we have $\mathbb{E}u_1 = pw + \delta(1-p)(1-w) \leqslant \mathbb{E}u_1^s$ and $\mathbb{E}u_2 = (1-p)w + \delta p(1-w) \leqslant \mathbb{E}u_2^s$. In case II, we have $\mathbb{E}u_2 = 1-w \leqslant \mathbb{E}u_2^s$, but $\mathbb{E}u_1 = w \geqslant \mathbb{E}u_1^s$. Nevertheless, their total utility is also equal to 1. The same thing happens with both pure equilibria in case III: one of the universities improves with signaling, and another one accordingly gets less, but their total utility is equal to one again. In the case of the mixed equilibrium, the total utility of the universities is equal to $\alpha\beta(w+\delta(1-w)) + \alpha(1-\beta) + (1-$

$\alpha)\beta + (1-\alpha)(1-\beta)(1-w+\delta w)$, which is less than one for all $\delta < 1$. All the above-mentioned may be formulated in the following proposition.

Proposition 4: *Implementing signaling is always beneficial for S_1 but not necessarily for U_1. In any case, the social welfare with signaling is higher or equal to the social welfare without signaling. Moreover, the total utility of students and universities also increases or at least stays the same.*

Thus, we can conclude that it may be not in the interests of top universities to implement signaling procedures for the job-market.

It is also worth mentioning that with signaling, the social welfare does not depend on w. Thus, the difference between students does not matter when they have opportunity to make signals and disclose their preferences.

Unfortunately, in the case of $n > 2$, signaling does not necessarily lead to stable outcomes in the first period and prevent delays. Another issue is that a student may not be interested in signaling truthfully. We will prove that for $n = 3$ after considering a general case of n agents from each side (see Section 7).

7 Generalization

Consider the case of n participants from each side. We will be interested in sufficient conditions for existence and nonexistence of assortative matching. Before formulating general results, we need to define the distribution (probability measure) over graduates' preferences. By "objective ranking", we will call the one with the highest probability. Matching with U_1 in this ranking gives n to any student, matching with U_2 gives $n-1$, ..., matching with U_n gives 1. Formally, for any $i \in \{1, \dots, n\}$

$$v_i(1) = n, \quad v_i(2) = n-1, \quad v_i(3) = n-2, \quad \dots \quad v_i(n-1) = 2, \quad v_i(n) = 1.$$

Let the probability of this default ranking $(n, n-1, n-2, \dots, 2, 1)$ be z. The number of all possible permutations is $n!$, and any permutation may be obtained from the default one as a composition of a finite number of specific transpositions that swap two adjacent elements of the sequence (Steinhaus-Johnson-Trotter algorithm[1] as one of possible ways). For example, (132) may be obtained from (321) this way: (321) $\rightarrow$ (312) $\rightarrow$ (132). Let each transposition of that kind diminish the probability of a new permutation by q. (Thus, in the previous example, $Pr\{(321)\} = z$, $Pr\{(312)\} = zq$, $Pr\{(132)\} = zq^2$.) If q is given, then z can be found from the probability measure condition (the sum must be equal to one).

[1] see, for example, Sedgewick (1977)

Also, for any $i \in \{1, \dots, n\}$, let

$$u_i(1) = n, \quad u_i(2) = n-1, \quad u_i(3) = n-2, \quad \dots \quad u_i(n-1) = 2, \quad u_i(n) = 1.$$

Figure 18 represents the model.

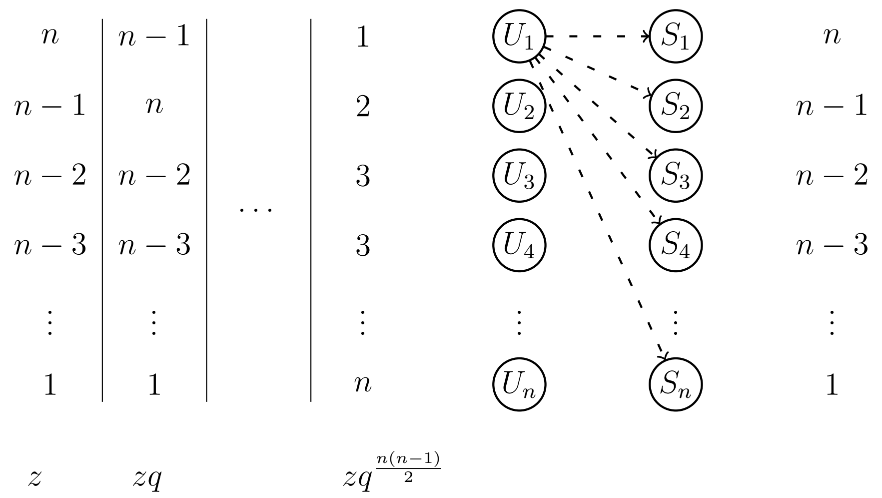

Figure 18: General case of n students and n universities.

Example 1 ($n = 2$): In this case, the probability of (21) is z and the probability of (12) is zq. Since $z + zq = 1$, we have $z = \frac{1}{1+q} = p$. Thus, as long as p changes from 1/2 (descrete uniform distribution: all universities are equal) to 1 (degenerate distribution: U_1 is always preferred), q changes from 1 to 0.

Example 2 ($n = 3$): In this case, $3! = 6$ different transmutations are possible. (321) has probability z. Transmutations (312) and (231) may be obtained from (321) by swapping of two adjacent elements and have probability zq each. Transmutations (213) and (132) may be obtained from transmutations (231) and (312) respectively with one more swapping of that kind. Thus, we need two consecutive swappings to obtain (213) and (132) from initial (321). Corresponding probability is zq^2. Finally, (123) has probability zq^3. Since $z + 2zq + 2zq^2 + zq^3 = 1$, we have $z = \frac{1}{(1+q)(1+q+q^2)}$, and the probability measure is defined (see Table 2):

Allocation	3 2 1	3 1 2	2 3 1
Probability	$\frac{1}{(1+q)(1+q+q^2)}$	$\frac{q}{(1+q)(1+q+q^2)}$	$\frac{q}{(1+q)(1+q+q^2)}$
Allocation	2 1 3	1 3 2	1 2 3
Probability	$\frac{q^2}{(1+q)(1+q+q^2)}$	$\frac{q^2}{(1+q)(1+q+q^2)}$	$\frac{q^3}{(1+q)(1+q+q^2)}$

Table 2: The probability measure of all possible preference orders in case $n = 3$.

For different q, we have different distributions: from the degenerate distribution to the uniform one (see Table 3). For example, the distribution described in Proposition 7 may be obtained when $\frac{1}{(1+q)(1+q+q^2)} = 1-\varepsilon$. Thus, for $\varepsilon = 0.2$ we get $q \approx 0.112$ and for $\varepsilon = 0.1$ we get $q \approx 0.053$.

q	**Distribution**
0	$(1, 0, 0, 0, 0, 0)$
0.2	$(0.67, 0.13, 0.13, 0.03, 0.03, 0.01)$
0.5	$(0.38, 0.19, 0.19, 0.095, 0.095, 0.05)$
0.8	$(0.23, 0.18, 0.18, 0.15, 0.15, 0.12)$
1	$(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$

Table 3: Distributions for different q in case $n = 3$.

The next two statements guarantee existence and nonexistence of assortative matching in this general case.

Proposition 5: For any n, if market participants are sufficiently impatient and if the likelihood that students' preferences match the objective ranking is sufficiently high, then assortative matching is an equilibrium. Namely, for any n and $\delta \leqslant 1/n$ there exists $\varepsilon > 0$, s.t. for any $q \leqslant \varepsilon$ assortative matching is an equilibrium.

Proof: First note that students necessarily accept any offer in the first period. Indeed, even if the worst student rejects an offer from the university she likes the least ($v_1 = 1$) in order to be matched with her most preferable university ($v_1 = n$) in the second period, the discount factor $\delta \leqslant 1/n$ does not allow her to get a better payoff. Second, we can find sufficiently small q, such that universities do not want to deviate: nobody wants to compete with a better university if the probability to get higher in the ranking is lower than the difference in payoffs. ■

Proposition 6: For any n, if the likelihood that students' preferences match the objective ranking is sufficiently low, then assortative matching is not an equilibrium. Namely, for any n and $\delta > 0$ there exists $\varepsilon > 0$, s.t. for any $q \geqslant 1 - \varepsilon$ assortative matching is not an equilibrium.

Proof: If q is close enough to 1, U_n has almost the same chances to be chosen by the best student as has U_1. Thus, if U_n deviates and competes with U_1 for the best student, its payoff even under smallest possible δ will be close to $n/2$. For any $n \geqslant 3$ it is larger than the highest possible payoff that this university can earn sticking to assortative matching. ■

Although those estimations are quite conservative, they show that no matter what the market size is, assortative matching equilibrium is either guaranteed or cannot exist under certain values of parameters.

Another question is whether signaling works for higher n the same way as it deals with delays and coordination failures in the case of two agents on each side. In general, the answer is negative. Also, in the case of $n > 2$, a student may not be interested in signaling truthfully. We will prove that for $n = 3$ after considering a general case of n agents from each side (see Section 7).

Proposition 7: *In case $n \geqslant 3$, truthful signaling may not be an equilibrium strategy.*

Proof: It is enough to formulate a counterexample for $n = 3$. Let U_i and S_j be universities and students, respectively ($i, j \in \{1, 2, 3\}$). For any j, students' utility of matching the most preferable university is $v_j = 3$, the second best — $v_j = 2$, and the least preferable school delivers $v_j = 1$. Assume that the preference order $U_1 \succ U_2 \succ U_3$ has probability $1 - \varepsilon$. In particular, it implies that $Pr\{v_j(1) = 3\} \geqslant 1 - \varepsilon$ (because it contains the probability of $U_1 \succ U_3 \succ U_2$). Thus, in case of truthful signaling, with probability higher than $(1 - \varepsilon)^3$ all signals go to U_1.

Since all the above-mentioned information is a common knowledge, U_2 and U_3 understand that with overwhelming probability U_1 will match S_1 and they have to compete for S_2 and S_3. Hence, with probability $(1 - \varepsilon)^3$ we can consider that problem as a case of two universities U_2 and U_3 and two students S_2 and S_3 analogous to what was described in sections 2–4 with $p \geqslant 1 - \varepsilon$ (since $U_2 \succ U_3$ is already contained in $U_1 \succ U_2 \succ U_3$) and $w = 2/3$ (normalization of utilities 2 and 1). According to figure 14, for these p and under any $\delta \leqslant 1 - 2\varepsilon$ we fall into case II. Thus, should truthful signaling be an equilibrium, U_2 makes an offer to S_2 and U_3 makes an offer to S_3, and both students accept.

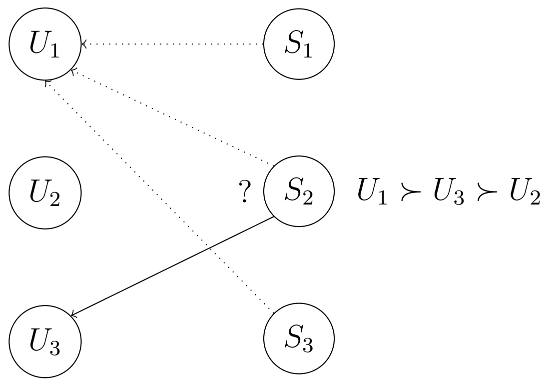

Figure 19: Sometimes it is more profitable for S_2 to deviate from the truthful signal (dotted lines) to her second best (solid line).

Now assume that S_2 has a preference order $U_1 \succ U_3 \succ U_2$. Is it still more profitable for S_2 to signal truthfully, or should the second student deviate and make a signal to its second best U_3 (see Fig. 19)? The expected utility of S_2 in the first case is not higher than $\mathbb{E}v_2 = (1-\varepsilon)^3 \cdot 1 + (1-(1-\varepsilon)^3) \cdot 3$. However, in the second case, U_3 gets a signal from S_2 and tries to "steal" her from U_2. There is still a chance that S_2 will receive an offer from U_1 (it may happen in the case when, for example, S_1 prefers U_2 over U_1: then U_1 does not get any signal from S_1 and goes down for U_2), but the probability of this event is lower than $1-\varepsilon$. The resulting utility from this deviation is at least $\mathbb{E}v_2 = 2$ which is higher than $(1-\varepsilon)^3 \cdot 1 + (1-(1-\varepsilon)^3) \cdot 3$ under any $\varepsilon < 1 - \frac{1}{\sqrt[3]{2}} \approx 0.2$. Hence, in this case truthful signaling is not consistent with an equilibrium path. ■

This example shows exactly the behavior of some students on a job market. If a person understands that she is definitely not one of the best students in the cohort, there is no reason for her to signal to top universities (even if it is still her first best). Instead of that, it might be better to spend the valuable signal on something more credible, especially if this choice is unexpected to other market participants.

At the same time, we notice that in this particular case deviation of student S_2 may not lead to a stable matching outcome. Assuming that S_1 prefers U_1 the most, we can see that to reach stability in the first round, U_2 must make an offer to S_3. Depending on the value of δ and the signal preferences of S_3, this may be not the most profitable strategy for U_2 in expectation.

Thus, implementing signaling does not necessarily eliminate delays and coordination failures. In the next section, we consider another mechanism that helps us to reach that goal even for $n > 2$.

8 Incentivized Immediate Response

In the benchmark model, universities make an offer and have to wait until the next round for an answer, even if the answer is negative (reject). For example, this inefficiency becomes evident on the market of prospective PhD students in economics in the US. Universities make offers in February–March, but students are not obliged to give an answer until the 15th of April. Even if a student has already received a better offer, she does not have any incentives to report back to all other universities that have also offered her a position. Should a university get a negative response (reject) early, it could have saved time to make another offer to its second best (in the above-mentioned example, much earlier than April, 15). Mechanisms of forcing students to report immediately may be different. Probably, the most obvious one is a (partial) return of the application fee. The purpose of this section is not to discuss possible mechanisms but to understand how implementing them would change our model. In particular, we are interested again in optimizing our matching process in order to avoid delays and coordination failures.

Apparently, students may be incentivized to respond (report) immediately if it does not decrease their payoff. It may happen in two cases:

- having received two or more offers, they *immediately* reject all underwhelming ones
- having received an offer from their first best, they *immediately* accept it.

Assume that *immediately* here means such an infinitesimal period, so the corresponding university has time to make another offer to its second best within the same period of time. We call this kind of reports within a round *immediate response.*

Proposition 9: With immediate response, all the participants necessarily match in the first round in equilibrium. This matching is always stable, and the only equilibrium here is full competition where all the universities make an offer to the best student.

Proof: Assume two different universities U_i and U_j make offers to different students S_k and S_l, respectively ($k < l$). Then U_j always wants to deviate to S_k: in this case, it either gets a better match or receives immediate reject (if S_k prefers U_i) and returns to S_l within the same round without any loss of payoff. Thus, we can conclude that in equilibrium all the universities must make an offer to the same student S_k.

Let's prove that $k = 1$. Assume by contradiction that $k > 1$. Then U_1 would like to deviate. Indeed, if U_1 makes an offer to S_1, then other universities that have been immediately rejected by S_k make an offer to S_1 as well. If S_1 prefers U_1, the last one gets utility n. If S_1 prefers any other university from those that made an offer, U_1 returns to S_k within a round and does not lose any payoff. The trickiest part is if U_1 is the second best for S_1 and its first best U_i in its turn is the second best of S_k. In this case, it is equivalent to a framework of two universities and two students where students get offers from the underwhelming universities. No matter if δ is high or low, this deviation of U_1 gives it higher utility then (see sections 3 and 4).

Thus, we have proved that all the universities make an offer to S_1 first. After one of them gets matched, the process repeats again: remaining $n-1$ universities make an offer to the best remaining student, namely S_2. The procedure continues until S_n gets matched with the last remaining university.

Stability of this matching may be supported by the fact that this procedure is equivalent to "deferred-acceptance" algorithm, presented by Gale and Shapley (1962). Moreover, this matching is optimal for students. Indeed, we may think of our process the following way. Students submit their offers first, and the universities that get more than one offer reject all except the best one. Students who get rejected, submit offers to their second best universities, and so on. The result is obviously the same: the best student gets what she wants, S_2 gets what is left after S_1, S_3 gets the best university except those that get matched with either S_1 or S_2, etc. ■

Corollary: The market behavior with incentivized immediate response is equal to the benchmark model with $\delta = 1$.

Proof: If $\delta = 1$, each university makes an offer to the best student first, since delay does not matter anymore. Thus, reject in each round of the benchmark model is equivalent to immediate reject within a round. ■

We can see that immediate response is another way to avoid delays and coordination failures. We get a stable matching that is optimal for students but not for universities. As in the case of signaling, good universities may not be interested in implementing that procedure. However, at least in case $n = 2$, the social welfare of both universities and students gets higher or equal with immediate response. The corresponding utilities are absolutely equivalent to ones with signaling.

9 Conclusion

In this paper, we consider a decentralized two-period matching model, where preferences of one side of the market (universities) are common knowledge and preferences of the other side (students) are private information. Even in a simplest framework with only two participants from each side of the market, the behavior of the agents happens to be quite sophisticated and may result in either delays or coordination failures. We obtain necessary and sufficient conditions on parameters of the model that lead to different strategies of both students and universities. The result called "threat of reject" is formulated: under some values of parameters, some participants of the market have better expected payoffs if they are forced to accept an offer in the first period. This problem may be solved by signaling before the first period or immediate response. We show that both these improvements completely eliminate any delays and coordination failure issues (in case of signaling, unfortunately, only in the simplest case). However, although implementing signaling and immediate response is always beneficial for the best student and increases the social welfare, it is not necessarily beneficial for the best university.

We also obtain some general results. In the case of n universities and n students, we derive sufficient conditions on parameters for existence and nonexistence of assortative matching. We show that in the case of $n \geqslant 3$ participants from each side, students may not be interested in truthful signaling.

For future studies, it might be useful to examine the signaling mechanism in a general case for better understanding to what extent it helps us to avoid delays and coordination failures. Also, it would be interesting to apply quotas to this model when (some) universities may need to fulfill more than one position and, thus, submit more than one offer.

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