# Application Costs as a Screening Instrument in Decentralized Matching

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**Abstract**

We consider decentralized matching in a two-sided market of firms and workers with application costs and limited budgets. Workers choose whether they should take the risk of applying to a higher-ranked firm (with some probability of rejection) or make a safe choice. We show that application costs set by firms with uncertain capacity may be treated as a screening instrument in order to attract only strong applicants or avoid the competition. Surprisingly, competition may lead to increasing costs: low-ranked firms may even prefer implementing highest possible costs in equilibrium. We provide economic intuition behind this paradox and find necessary and sufficient conditions that lead to this result.

Key words: *decentralized matching, non-cooperative game theory, school choice, screening*

JEL: *D47, C78.*

# 1 Introduction

## 1.1 Overview

The majority of theoretical literature on two-sided matching markets is devoted to centralized markets. Although it is necessarily dictated by a vast number of real life problems (e.g., school allocations, the medical residency match, kidney exchange, etc.), there are many examples of markets that are not (fully) centralized (for example, the market for junior economists, paper submission to journals, dating sites, and so on). Thus, theoretical

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understanding of the outcomes generated by decentralized markets becomes crucial for institutional design.

The general approach to matching markets is based on the pioneer work of Gale & Shapley (1962) and is thoroughly summarized in Roth & Sotomayor (1990) monograph. Another great survey analyzing, among others, different matching theory applications in labor economics, macroeconomics, and monetary theory was brought by Chade, Eeckhout, and Smith (2017). However, the canonical notion of stability that is used in the vast majority of papers becomes inapplicable if we need to consider matching frictions. They may include incomplete information, dynamics, transfers, etc. In this paper, we introduce a decentralized matching model with application costs on one side of the market and limited budgets and incomplete information on the other side. To find optimal matchings, we take an approach based on non-cooperative game theory. Namely, we describe a matching process as a game where all agents have their own strategies. Thus, we can use Nash equilibrium as a concept that allows us to successfully distinguish "good" and "bad" matchings. On one hand, it significantly complicates the model. On the other hand, we obtain more nuanced results that take into account matching frictions and get better understanding of the matching process itself.

We consider a two-sided market of firms and workers (institutions and applicants), where firms simultaneously choose application costs in order to maximize their profits from the entire pool of applicants. Workers in turn make a decision where to apply facing these costs and their budget constraints. We show that application costs set by institutions with uncertain capacity may be treated as a screening instrument in order to avoid the competition, attract only strong workers, and regulate a number of applicants. Perhaps surprisingly, competition may lead to increasing costs: low-ranked firms may even prefer implementing highest possible costs. We provide economic intuition behind this paradox and find necessary and sufficient conditions that lead to this result and many other equilibria of interest. Also, the case when firms can change their threshold of acceptance is considered. We find the optimal value of this threshold in the simplest case of two firms.

Probably the most natural matching friction that goes along with incomplete information is dynamics. This may include early admissions (Avery and Levin, 2010), job-market signaling (Coles, Kushnir, and Niederle, 2013), or exploding offers (Pan, 2018), among others. All these papers use Nash equilibrium concept to distinguish "optimal" matchings. Although the notion of stability was successfully reconsidered for dynamic settings (Doval 2021, Ho 2021), decentralized markets with uncertainties dictate new restrictions.

Focusing on the matching literature that analyzes incomplete information along with application costs and/or admission thresholds, we should mention several papers. Avery and Levin (2010) introduce a two-period model with two universities and a continuum of students (which is similar to our framework). Universities may change their admission thresholds during both periods to sort or attract students. Equilibria they found show that both universities appear to favor early applicants and that early admission process may benefit lower-ranked schools. Another paper by Nguyen, Peters, and Poitevin (2017) considers a model of two universities and two students with application costs and incomplete information about students' types. The authors capture a well-known effect: lower-ranked schools don't necessarily make an offer to the best student among those who applied. Arnosti, Johari, and Kanoria (2019) consider a dynamic model (agents arrive and depart over time) with application costs on both sides of the market: it's costly to apply, and it's costly to screen applicants. They show that there may be inefficiency on either of the sides. On one hand, employers may have screened the same applicant who matches only one of them afterwards (wasted effort). On the other hand, some applications may remain unscreened (congestion). He and Magnac (2019) analyze empirically how application costs may help to avoid this congestion. They conduct an experiment with admissions to master's programs at the Toulouse School of Economics and show that low application costs effectively reduce congestion without harming match quality.

Probably the closest paper to ours is the one by Chades, Lewis, and Smith (2014, CLS). They formulate the college admissions problem for two ranked schools with fixed capacities and costly applications. While having much in common, our model is significantly different from theirs in many important aspects. First, we assume an unlimited capacity for firms. Although a limited capacity is more natural in some student-to-school matching environments, there are examples such as online platforms where firms do not face a capacity constraint and would like to match with as many individuals as possible. Also, the number of enrolled workers may be successfully regulated by setting application costs and the quality threshold in admissions. A perfect illustration of this statement is PhD enrollment where sizes of cohorts may slightly fluctuate from one year to another. Second, we use application costs as a part of firms' strategy sets, while in CLS application costs are fixed. Again, fixed costs make perfect sense for college admission but give a lack of flexibility in many other applications. For example, in transport economics firms need to decide where to choose the location of their store inside the city limits: different locations mean different transportation costs. Finally, in our model workers make their decisions already knowing the admission threshold and application costs, while in CLS firms and workers simultaneously choose the threshold and where to apply, respectively. Therefore, in our framework applicants have full information about the institutions' admission criteria.

This captures a setting where students already know (or have a very good idea of) what it takes to get admitted to the high-quality schools.

Because of these distinctions, some results in two discussed models are completely different. For example, in our model better firms still get better workers. Therefore, we do not observe this violation of monotonicity (when lower-skilled students apply to higher-ranked universities) which is implied by limited capacity. However, many other findings of CLS also take place in our model. Probably the most important one is that college admission standards do not necessarily reflect their quality: the lower-ranked institution may optimally implement higher admission threshold (in CLS) or higher application costs (in our model). Thus, competition may surprisingly lead to increasing prices.

Another important feature of our model is a stochastic independence of worker's admission prospects across firms. Here we follow the approach of Chade and Smith (2006). In our model, a worker can observe her own type, and the noise is independent across firms. However, assume that applicant does not know her type. For example, a student needs to decide upon her college application portfolio before she gets to know the results of her high school test (see Nageeb Ali and Shorrer, 2021). In this case, her probabilities of being enrolled to higher-ranked firms will be correlated. Rees-Jones, Shorrer, and Tergiman (2019) provide an experiment showing that participants fail to account for this correlation. Instead of applying aggressively to two good schools, a rational student should diversify her strategy by applying to both good and safe schools, but the experiment outcome demonstrates the opposite.

## 1.2 Motivating Example

Let's start with a simple motivating example (see Fig. 1). There is a continuum pool of workers $(W_\theta)_{\theta\in[0,1]}$ uniformly distributed on $[0,1]$ (thus, $\theta \sim \mathbf{U}_{[0,1]}$). Also, we have three firms $V_2$, $V_1$, and $V_0$ with unlimited capacity. First two of them are considered higher-ranked, so the utility of getting there for any worker is 3. The payoff from being matched with $V_0$ is 1. The probability that worker $W_\theta$ will be hired by a higher-ranked firm is equal to worker's type $\theta$, and it is independent among $V_1$ and $V_2$. The lower-ranked firm is ready to accept $W_\theta$ with probability 1. We can call $V_0$ a safe choice then, and $V_1$ and $V_2$ are risky choices.

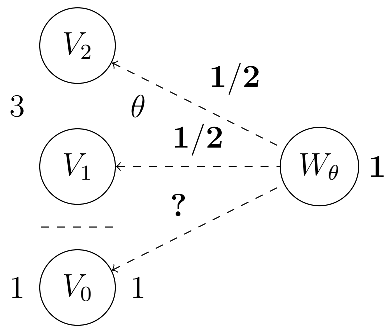


Figure 1: What application fee should be implemented by $V_0$?

Each worker knows her own type before making the decision. The safe firm values the worker with utility $u_0(\theta) = \theta$. Thus, the more capable the worker, the higher her chances to be hired by one of the higher-ranked firms and the more valuable for $V_0$ she is.

Another restriction is that the budget of each $W_\theta$ is equal to 1, and $V_1$ and $V_2$ set their application costs as a half of the budget each. Thus, the question is: *what application fee must be implemented by* $V_0$? Should it be less than the one set by the higher-ranked firms?

For example, if the cost for $V_0$ is zero, a worker will have a chance to apply to all three firms. If it's higher than zero but lower than 1/2, a worker faces the hard decision choosing between applying to two higher-ranked firms or diversifying her application portfolio. Finally, $V_0$ may force a worker to choose between taking solely her safe option and applying to two higher-ranked firms by setting the application fees higher than 1/2. In the first case, all workers will apply to $V_0$ for free. In the second case, the marginal utility of making the risky choice is $3\theta$, and the marginal utility of making the safe choice is 1. Thus, a risk-neutral worker applies to two higher-ranked firms if and only if $\theta \geqslant 1/3$. Finally, in the third case, a worker chooses between being accepted to at least one of two higher-ranked firms (and getting $3(1-(1-\theta)^2)$) and $V_0$ (and getting 1). She would choose to apply solely to $V_0$ if and only if $\theta < 1 - \sqrt{2/3}$. This paragraph can be summarized mathematically (here $c_0$ is an application fee set by $V_0$, and $U_0$ is an aggregate utility of $V_0$ for the entire pool of workers $(W_\theta)_{\theta\in[0,1]}$):

$$
\begin{aligned}
U_0 &= \int_0^1 \theta(1-\theta)^2 d\theta = 0.083 && \text{if } c_0 = 0,\\
U_0 &= \int_0^{1/3} \theta(1-\theta) d\theta = 0.043 && \text{if } 0 < c_0 \leqslant 1/2,\\
U_0 &= \int_0^{1-\sqrt{2/3}} \theta d\theta = 0.017 && \text{if } 1/2 < c_0 \leqslant 1.
\end{aligned}
$$

In the first integral, we can see that everyone applies to $V_0$, but only the workers who get hired neither by $V_1$ nor by $V_2$ (probability of this event is $(1-\theta)^2$) accept an offer from $V_0$. In the second integral, two thirds of the workers with higher type do not even apply to $V_0$. The lower half applies to $V_0$ and either $V_1$ or $V_2$. Thus, they accept an offer from $V_0$ with probability $1-\theta$. Finally, in the third integral only the workers with type $\theta < 1-\sqrt{2/3}$ are going to apply to $V_0$. They all accept an offer from $V_0$, since it is their only application.

This example demonstrates that the safer firm needs to implement zero costs in order to attract better workers and get more profit. But what if the difference between firms is not so drastic? For example, assume now that $a = 3/2$ and calculate the aggregate utility in this case.

$$
\begin{aligned}
U_0 &= \int_0^1 \theta(1-\theta)^2 d\theta = 0.083 && \text{if } c_0 = 0,\\
U_0 &= \int_0^{2/3} \theta(1-\theta) d\theta = 0.123 && \text{if } 0 < c_0 \leqslant 1/2,\\
U_0 &= \int_0^{1-1/\sqrt{3}} \theta d\theta = 0.089 && \text{if } 1/2 < c_0 \leqslant 1.
\end{aligned}
$$

We can see that, maybe against our expectations, $V_0$ now should implement the same fees as the higher-ranked firms forcing applicants to make the unpleasant decision and taking a substantial part of the pool away from $V_1$ and $V_2$. Moreover, if we continue to decrease $a$, we will realize that starting from $a = 1.26$ the most profitable strategy for $V_0$ will be setting the highest possible costs equal to 1. We will call this strategy restrictive and, should it be a part of the equilibrium, we will call this situation on the market Restrictive Equilibrium. Indeed, the lower-ranked firm makes a job offer to a worker. However, should she reject it, she would be able to apply to all the higher-ranked firms, but there will be no safe option anymore.

We can observe the similar picture analyzing the quality of workers. The more high-skilled workers are on the market, the less incentives the safer firm has to set higher costs, and vice versa.

The most natural (and inspiring) example of this model is the market of prospective PhD students. Here firms are universities, and workers are PhD applicants. More precisely, we are interested in its difference between North American and European schools. Namely, the majority of European schools do not impose any application fees (we may consider paperwork as some negligible cost though), while the situation in the US is actually the opposite. Most of American universities charge \$70–\$120 for considering an application. Thus, we may say that our model is inspired (but not limited) by two markets of prospective PhD students in North American and European schools. We find conditions for equilibria when costs are negligible (Inclusive, or European, equilibrium), costs are a substantial part of the budget (Selective, or American, equilibrium), costs imply Restrictive equilibrium (the lower-ranked university tries to "steal" the higher-ranked student), in addition to other equilibria. Also, application costs may reflect time that a student spends on preparing their documents. For example, some schools may require a standard application package, and others may ask for some unique files or documents that take time and/or effort to prepare (e.g., essays or additional recommendations).

Another noticeable setting is a model of opening firms (stores) inside the city limits. Here the budget of a customer is time that she is willing to spend on shopping. Depending on customer needs (or willingness to spend money) $\theta$, she might choose a store with worse quality but better chances to find the good she needs or vice versa. In the example above, this low-quality store should choose whether it wants to open right in the city center (and provide zero transportation costs for citizens) or somewhere on the outskirts.

Publications in scientific journals is another great example that satisfies the model if we do not consider dynamics or keep the discount factor small enough to focus only on the first round of submissions. Depending on quality of the paper $\theta$, an author may send the paper to a mediocre but safe journal or try to publish higher with some probability of reject. We can assume that the market here is decentralized and check if there is an equilibrium which describes exactly what happens with this market in real life. Our findings tell us that the equilibrium that forces authors to send their manuscripts only to one journal at the time ($c_i = 1$ for all $i$) is approachable when the distribution of applicants is skewed towards low types (the majority of papers are relatively bad).

Also, with some adjustments, our model may be applied to online platforms (selling goods and consumer search), venture capital investments, and online dating sites, among others.

We continue with the general description of the model in Section 2 and consider the case of two universities in Section 3. For them, we will describe all the eqilibria and find an optimal threshold for a higher-ranked firm. Section 4 is devoted to a richer case of three firms when the probability of being accepted by a higher-ranked firm is equal to the type of a worker with substantial examples provided. Section 5 contains some general results. Proofs and complex calculations are taken out to Appendix.

## 2 The Model

Consider a two sided-market with $m+1$ participants on one side and a continuum of participants on another side. These participants form set $V = \{V_1, V_2, \ldots, V_m, V_0\}$ of firms (institutions) and set $W = \{W_\theta\}_{\theta \in [0,1]}$ of workers (applicants), respectively. All applicants have an equal budget (endowment) that we normalize to 1. Institution $V_i$, $i \in \{1, 2, \ldots, m, 0\}$ introduces application cost $c_i \geqslant 0$, and workers apply to firms paying those costs up to their budget constraints. Formally,

$$\sum_{i=1,2,\ldots,m,0} c_i \; \mathrm{I}\{W_\theta \text{ applies to } V_i\} \leqslant 1 \tag{1}$$

for any $\theta \in [0,1]$. Here, $\mathrm{I}\{W_\theta \text{ applies to } V_i\}$ is equal to 1 if $W_\theta$ applies to $V_i$, and zero otherwise.

Type $\theta$ of each worker $W_\theta$ is a random variable on segment $[0,1]$ with cumulative distribution function $F$. This type is entirely disclosed only to a worker herself. For any $i$, firms $V_i$ observe noisy types $x_i = \theta + \varepsilon_i$ where $\varepsilon_i$ are mutually independent and identically distributed symmetric random variables with cdf $G$. We assume that $G$ is absolutely continuous. Subindexes may be omitted if it does not result in any confusion.

A worker applying to $V_0$ gets accepted with probability 1. A worker applying to $V_i$ ($i \in \{1, \ldots, m\}$) gets accepted if her observed type $x_i$ is higher than $\bar{x}_i$. These $\bar{x}_i$ are common knowledge and may either be given exogenously or defined by firms in advance. In any case, vector $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ is disclosed to both workers (before they make their decision about where to apply) and firms (before they implement costs). Let $p$ be the probability of being enrolled by $V_i$ ($i \in \{1, \ldots, m\}$). Then

$$p = Pr\{x \geqslant \bar{x}\} = Pr\{\varepsilon \geqslant \bar{x} - \theta\} = 1 - G(\bar{x} - \theta). \tag{2}$$

All firms value workers depending on their type $\theta$. Let nondecreasing functions $u_i(\theta)$ be the utilities of firms $V_i$, $i \in \{1, \ldots, m, 0\}$. (We also assume that $u_0(0) \geqslant 0$.) Then aggregate expected utility functions for firms are

$$U_i = \int_0^1 u_i(\theta) \; Pr\{W_\theta \text{ matches } V_i\} dF(\theta) \qquad (i = 1, \ldots, m, 0).$$

Here, by "$W_\theta$ matches $V_i$" we mean that $W_\theta$ had applied to $V_i$, was made an offer, and accepted it.

Finally, let $w(i)$ be a worker's utility of matching with $V_i$: $w(0) = 1$ and $w(i) = a > 1$ if $i \in \{1, \ldots, m\}$. Thus, we can see that firms $V_1, \ldots, V_m$ are higher-ranked than $V_0$, and it is harder to get there. Lower-ranked $V_0$ may be called a safe choice. If a worker gets hired by both higher-ranked and lower-ranked firms, she always chooses the first one. Also, we assume that if an applicant gets enrolled to $k$ of $m$ higher-ranked firms, she chooses one of them with equal probability $1/k$. The total welfare of all workers will be denoted as $W$.

Now we are ready to describe the entire game. First, thresholds $\bar{x}$ get determined (either by "nature" or by institutions themselves). Then firms $V_1, V_2, \ldots, V_m$, $V_0$ independently set their nonnegative costs $c_1, c_2, \ldots, c_m, c_0$ in order to maximize their expected utilities $U_i$. At the next step, workers apply according to their types and budget constraint (1) in order to maximize their own expected payoffs $w$. Unmatched workers and firms receive zero utility.

Let's describe our game formally. For simplicity, consider $\bar{x}$ as given by nature. Thus, setting a threshold is not a part of a firm's strategy profile. We define the strategy set of the game as

$$\left(A_1, \ldots, A_m, A_0, (B_\theta(c))_{\theta\in[0,1], c\in([0,1]\cup\{1+\delta\})^{m+1}}\right).$$

Here, a strategy set for firm $i$ is denoted by $A_i = [0,1] \cup \{1+\delta\}$ $(i \in \{1, \ldots, m, 0\})$. A strategy set for worker $\theta$ in each information set defined by firms' actions $c = (c_1, \ldots, c_m, c_0) \in (A_1, \ldots, A_m, A_0)$ is

$$B_\theta(c) = (\mathrm{I}\{W_\theta \text{ applies to } V_i\})_{i=0}^m, \quad \text{such that} \sum_{i=0}^m \mathrm{I}\{W_\theta \text{ applies to } V_i\} c_i \leqslant 1.$$

Again, $\mathrm{I}\{\cdot\}$ is an indicator here. Value $1+\delta$ in $A_i$ describes the case when $V_i$ does not want to enter the market and remains with zero payoff (we could use any $c_i > 1$ instead).

Should firms be allowed to choose $\bar{x}_i \in \bar{X}_i$, sets $\bar{X}_i \in \mathbb{R}$ $(i \in \{1, \ldots, m\})$ will be included in the strategy set of the game:

$$\left(\bar{X}_1, \ldots, \bar{X}_m, (A_1(\bar{x}))_{\bar{x}}, \ldots, (A_m(\bar{x}))_{\bar{x}}, (A_0(\bar{x}))_{\bar{x}}, (B_\theta(c, \bar{x}))_{\theta, c, \bar{x}}\right).$$

We can see that now firm and workers need to define their strategies for any possible value of $\bar{x}$.

Note that technically this market may be called the game only from the firms' perspective. Workers make decisions in order to maximize their own profits, and these decisions do not depend on decisions of other applicants. Nevertheless, we include all the possible sets of workers' actions for completeness.

Throughout the entire paper, we consider only pure strategy subgame perfect Nash equilibria. Moreover, from the whole set of Nash equilibria (NE) that deliver the same payoffs to all the participants, we will distinguish a particular NE (and call it cNE) with maximum and symmetric application fees for firms $V_1, \dots, V_m, V_0$. Thus, we implicitly assume a sort of lexicographic preferences for firms: first finding sets of Nash equilibria using payoffs that do not value costs; then finding contraction of NE in each set by maximizing costs for the firms. The next definition holds.

**Definition.** Let $C$ be a set of pure strategy Nash equilibria for $V_1, V_2, \dots, V_m, V_0$ where for each element of this set (i.e., for each strategy profile $c \in C$) utilities of all the universities do not change. Let $\mathcal{C}$ be a collection of all these sets. Precisely,

$$\mathcal{C} = \{C \in (A_1, \dots, A_m, A_0) : \forall i \in \overline{0, m} \ \forall c, c' \in C \ (c \text{ is NE}, u_i(c) = u_i(c'))\}.$$

Define cNE as an injective function cNE : $\mathcal{C} \to (A_1, \dots, A_m, A_0)$, such that cNE$(C) = c^*$ if and only if the next statements hold:

1. $c^* = (c_1^*, \dots, c_1^*, c_0^*) \in C$,
2. if $\exists c = (c_1, \dots, c_1, c_1) \in C$, then

$$c_1^* = c_0^* = \max_{c=(c_1,\dots,c_1)\in C} c_1,$$

otherwise

$$c_1^* = \max_{c=(c_1,\dots,c_1,c_0)\in C} c_1, \qquad c_0^* = \max_{c=(c_1^*,\dots,c_1^*,c_0)\in C} c_0.$$

Note that this definition is correct in the sense that corresponding $c$ exists and unique for any $C$.

If there is no confusion, we may omit subindices in utility functions for higher-ranked firms due to symmetry: for any $i = 1, \dots, m$, $u(p) = u_i(p)$, $U = U_i$.

Before considering different cases, let's return to our equation (2) that defines probability $p$ and formulate an important lemma.

**Lemma.** Probability $p$ of being hired by a higher-ranked firm is equal to type $\theta$ of a worker if and only if the threshold is equal to $1/2$ and the noise is uniformly distributed on segment $[-1/2, 1/2]$: $\bar{x} = 1/2$, $\varepsilon \sim \mathbf{U}_{[-1/2,1/2]}$.

**Proof.** Let $p = \theta$. Then from (2) we have $G(\bar{x} - \theta) = 1 - \theta$ for any $\theta \in [0, 1]$. Since noise $\varepsilon$ is symmetric, we have $G(x) = 1 - G(-x)$ for any $x$. Combining the last two equations, we obtain

$$G(\theta - \bar{x}) = 1 - G(\bar{x} - \theta) = 1 - (1 - \theta) = \theta.$$

For $\theta = 1/2$, we have $G(1/2 - \bar{x}) = 1/2$ and $G(\bar{x} - 1/2) = 1/2$. Since $G$ is absolutely continuous and, therefore, strictly increasing, we necessarily have $1/2 - \bar{x} = \bar{x} - 1/2$ which implies $\bar{x} = 1/2$. Thus, $G(1/2 - \theta) = 1 - \theta$. Considering $1/2 - \theta = t$, we get $G(t) = 1/2 + t$ which is exactly the cdf of $\mathbf{U}_{[-1/2,1/2]}$.

The opposite direction of the proof is obvious. ■

Thus, we can conclude that the type of a worker may be equal to the probability of being accepted only in one possible scenario. We will consider this particular case later for $m \geqslant 2$. Meanwhile, we will focus on the simplest case $m = 1$ and examine behavior of agents under different values of parameters.

# 3 Case of two firms: $m = 1$

We start with the simplest case of only two institutions: higher-ranked but risky firm vs. lower-ranked but safe firm.

## 3.1 General case and Equilibria

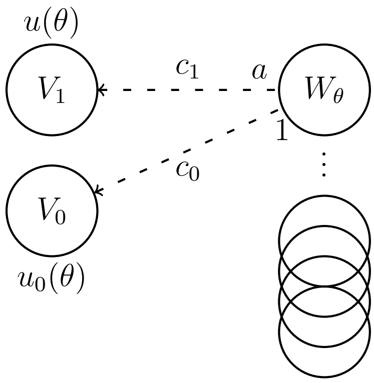


Figure 2: Case of two firms.

The game is drawn on Fig 2. The main concern of a worker under given costs $c$ is whether she should choose a higher-ranked firm (choice with uncertainty) or a lower-ranked one (safe choice). The marginal utility of applying to $V_1$ is $ap = a(1 - G(\bar{x} - \theta))$. The marginal utility of applying to $V_0$ is 1. Therefore, should a worker of type $\theta$ choose between the risky and the safe choice, she chooses a higher-ranked firm if and only if $p > 1/a$. Let's define the pivotal value of $\theta$ by $\hat{\theta}$. Then $\hat{\theta}$ is the root of equation

$$G(\bar{x} - \hat{\theta}) = 1 - \frac{1}{a}. \tag{3}$$

Since $G$ is strictly monotone and for any $a > 1$ we have $0 < 1 - 1/a < 1$, $\hat{\theta}$ is always unique (if it exists).

The next proposition describes all the possible equilibria in this game.

**Proposition 1.** Let

$$U_0^1 = \int_0^1 u_0(\theta)G(\bar{x} - \theta)dF(\theta), \qquad U_0^2 = \int_0^{\hat{\theta}} u_0(\theta)dF(\theta),$$
$$U_1^1 = \int_0^1 u_1(\theta)(1 - G(\bar{x} - \theta))dF(\theta), \quad U_1^2 = \int_{\hat{\theta}}^1 u_1(\theta)(1 - G(\bar{x} - \theta))dF(\theta).$$

Then the game on Fig. 2 has the following set of (subgame perfect) Nash equilibria ($0 \leqslant c_1 \leqslant 1$, $0 \leqslant c_0 \leqslant 1$):

1. if $U_0^1 > U_0^2$, $U_1^1 > U_1^2$, $U_1^1 \geqslant 0$, then the NE set is $(c_1, c_0)$, $c_1 + c_0 \leqslant 1$ with cNE $(1/2, 1/2)$
2. if $U_0^1 > U_0^2$, $U_1^1 < U_1^2$, $U_1^1 \geqslant 0$, then the NE set is $(c_1, 0)$ with cNE $(1, 0)$
3. if $U_0^1 > U_0^2$, $U_1^1 < 0$, $U_1^2 \geqslant 0$, then the only NE is $(1 + \delta, 0)$
4. if $U_0^1 < U_0^2$, $U_1^1 < U_1^2$, $U_1^2 \geqslant 0$, then the NE set is $(c_1, c_0)$, $c_1 + c_0 > 1$ with cNE $(1, 1)$
5. if $U_0^1 < U_0^2$, $U_1^1 > U_1^2$, $U_1^1 \geqslant 0$, then the NE set is $(0, c_0)$ with cNE $(0, 1)$
6. if $U_1^1 < 0$, $U_1^2 < 0$, then the NE set is $(1 + \delta, c_0)$ with cNE $(1 + \delta, 1)$.

The proof is in Appendix. ■

The degenerate case 6 is probably the most obvious one. $V_1$ does not enter the market, and $V_0$ gets all the workers. Case 3 is a little bit different. $V_1$ would like to enter if it could take only workers with types above the value $\hat{\theta}$. However, $V_0$ is not interested in splitting the market this way and sets zero costs. Under these conditions (when workers can apply to both firms), it is not profitable for $V_1$ to enter.

Cases 2 and 5 are symmetric in the sense that one firm would like to split the market by setting high costs, so workers could not apply to both institutions. Thus, the only option for the other firm to fight for the entire pool of workers is to set zero costs.

Cases 1 and 4 are the most interesting ones. In case 1, both firms are interested in sharing the market. $V_1$ finds it more profitable to have applications from the entire pool of workers but not only from the top, and $V_0$ prefers to take applicants who get rejected by $V_1$ rather than to deal only with bad workers. We will call it Inclusive Equilibrium for higher $m$ (see the next chapter). In case 4, both firms would prefer to have their own pool of workers: $V_1$ would like to get rid of low-skilled workers' applications, and $V_0$ does not want to accept only workers who were rejected by $V_1$. This is Selective and Restrictive Equilibrium together (they will be separated for $m \geqslant 2$).

Before moving to examples, we will raise one important question. Assuming that the higher-ranked firm can choose $\bar{x}$ itself before setting the costs, what is going to be the optimal value of $\bar{x}$ for $V_1$? Let's answer this question for linear utility functions and uniformly distributed noise.

## 3.2 Optimal $\bar{x}$ for Linear Utility Functions

Consider $u_0(\theta) = \theta$ and $u_1(\theta) = \theta - \alpha$ $(0 \leqslant \alpha \leqslant 1)$. It means that the lower-ranked firm would be happy to hire any worker from the pool, and the higher-ranked firm gets profit only from workers with type $\theta > \alpha$. But does it mean that $V_1$ should set threshold $\bar{x}$ equal to $\alpha$ in the case of uniformly distributed type $\theta$? Consider the following example.

**Example 1.** Let the higher-ranked firm be twice more valuable than the lower-ranked one and the noise be distributed uniformly on segment $[-l, l]$, $l \leqslant 1/2$. Therefore, we have: $\theta \sim \mathbf{U}_{[0,1]}$, $a = 2$, $\varepsilon \sim \mathbf{U}_{[-l,l]}$. Then from (2) and (3) we obtain

$$p = \begin{cases} 0, & \theta < \bar{x} - l, \\ \frac{1}{2} - \frac{\bar{x} - \theta}{2l}, & \bar{x} - l \leqslant \theta \leqslant \bar{x} + l, \\ 1, & \theta > \bar{x} + l, \end{cases} \qquad \hat{\theta} = \bar{x}.$$

The optimal value of $\bar{x}$ for different $\alpha$ is defined in Table 1.

| $\alpha$ | $[0, l)$ | $[l, 1-l)$ | $[1-l, \frac{1-l+x^*}{2})$ | $[\frac{1-l+x^*}{2}, \frac{1+2lx^*-(x^*)^2}{2(1+l-x^*)})$ | $[\frac{1+2lx^*-(x^*)^2}{2(1+l-x^*)}, 1]$ |
|---|---|---|---|---|---|
| $\bar{x}$ | $2\alpha - l$ | $\alpha$ | $2\alpha - (1-l)$ | $x^*$ | $l + \alpha - \sqrt{l^2 + (1-\alpha)^2}$ |
| cNE | (1,0) | (1,0) | (1,0) | (1,1) | (1,1) |

Table 1: Optimal $\bar{x}$ for different $\alpha$ if $a = 2$, $F(\theta) = \theta$ $(0 < \theta < 1)$, $G(x) = \frac{l-x}{2l}$ $(-l < x < l)$

Here,

$$x^* = 2\sin\left(\frac{\pi}{6} + \frac{\arctan\left(\frac{\sqrt{l^3(2-l^3)}}{1-l^3}\right)}{3}\right) - l.$$

1. If $\bar{x} < l$, we have the only cNE $(1, 0)$ and the optimal value of $\bar{x}$ is $2\alpha - l$. This happens due to the bounded support of $\theta$ and a firm's updating. Indeed, let's assume a firm gets a signal that a type of a worker is $x = l/2$. Then the initial type $\theta$ must be uniformly distributed on segment $[0, 3l/2]$ (there are no values of $\theta$ below zero). Taking it into account and calculating the conditional mean, a firm obtains $\mathbb{E}(\theta \mid x = l/2) = 3l/4$. Therefore, if $\alpha = 3l/4$, then the optimal threshold should be $\bar{x} = l/2$. Setting a threshold below $l/2$, a firm hires workers that would deliver negative uitility in expectation. Setting a threshold above $l/2$, a firm loses some workers that would deliver expected profit. This optimal value exists only if $2\alpha - l < l$ which means $\alpha < l$. If $\alpha$ is small enough, the optimal $\bar{x} = 2\alpha - l$ will be less than $\alpha$.

2. On this segment, updating does not imply any shifts since the bounds cannot be reached. The only cNE here is still $(1, 0)$, and the optimal value of $\bar{x}$ is $\bar{x} = \alpha$. That happens when $l \leqslant \alpha \leqslant 1 - l$. If the range of the noise is infinitesimally small, we have $\bar{x} = \alpha$ almost everywhere.

3. On this segment, we have an equilibrium shift from $(1, 0)$ to $(1, 1)$. After this shift, the behavior changes completely.

Table 2 summarizes the findings for the extreme case of $l = 1/2$. Note that the only point where $\bar{x} = \alpha$ is $1/2$.

| $\alpha$ | $[0, 1/2)$ | $[1/2, 0.64)$ | $[0.64, 0.81)$ | $[0.81, 1]$ |
|---|---|---|---|---|
| $\bar{x}$ | $2\alpha - 1/2$ | $2\alpha - 1/2$ | $0.78$ | $\frac{1}{2} + \alpha - \frac{1}{2}\sqrt{5 - 8\alpha + 4\alpha^2}$ |

Table 2: Optimal $\bar{x}$ for different $\alpha$ if $l = 1/2$

### 3.3 Examples

**Example 2: Full disclosure.** Let $\varepsilon \equiv 0$. It means that firms can see the real type of any worker. Then all workers whose type $\theta$ is higher or equal than $\bar{x}$ would like to apply to $V_1$. The rest would apply to $V_0$. Consider two cases.

1. $U_1 = \int_{\bar{x}}^1 u_1(\theta) dF(\theta) < 0$. The higher-ranked firm doesn't get a positive payoff from its pool of applicants. Thus, the only equilibrium strategy for $V_1$ here is not to enter the market: $c_1 = 1 + \delta$. The lower-ranked firm takes all workers and receives aggregate utility $U_0 = \int_0^1 u_0(\theta) dF(\theta)$. Any $c_0 \in [0, 1]$ is an equilibrium strategy. cNE $= (1 + \delta, 1)$.

2. $U_1 = \int_{\bar{x}}^1 u_1(\theta) dF(\theta) \geqslant 0$. Now $V_1$ gets profit from participating. The payoff of $V_0$ is $U_0 = \int_0^{\bar{x}} u_0(\theta) dF(\theta)$. The whole square $[0, 1]^2$ ($(c_1, c_0) \in [0, 1]^2$) will be an equilibrium here. cNE $= (1, 1)$.

Now assume that $V_1$ can choose the threshold. Then the optimal value of $\bar{x}$ is $\bar{x}_o = \sup\{\theta : u_1(\theta) = 0\}$. For example, in case of linear utilities $u_1 = \theta - \alpha$,

$u_0 = \theta$ and uniform distribution of workers $F(\theta) = \theta$ ($\theta \in [0, 1]$), the higher-ranked firm gets $1 - \alpha$ workers with aggregate payoff $U_1 = \frac{1}{2}(1-\alpha)^2$, and the lower-ranked firm gets $\alpha$ workers with aggregate payoff $U_0 = \alpha^2/2$. The total welfare of workers is $W = \alpha + a(1 - \alpha)$.

We can see that in the case of complete information, application costs don't play its strategic role. Since workers know their application outcome, firms cannot affect workers' decisions by changing the costs. However, the situation changes completely if the noise becomes substantial.

**Example 3.** Let $\varepsilon \sim \mathbf{U}_{[-1/2,1/2]}$, $\bar{x} = 1/2$, $u_1 = \theta - 1/2$, $u_0 = \theta$, $F(\theta) = \theta$ ($\theta \in [0, 1]$). Calculating the expressions from Proposition 1, we get

$$U_0^1 = \int_0^1 \theta(1-\theta)d\theta = \frac{1}{6}, \qquad U_0^2 = \int_0^{1/a} \theta d\theta = \frac{1}{2a^2},$$
$$U_1^1 = \int_0^1 (\theta - \frac{1}{2})\theta d\theta = \frac{1}{12}, \quad U_1^2 = \int_{1/a}^1 (\theta - \frac{1}{2})\theta d\theta = \frac{1}{12} + \frac{1}{4a^2} - \frac{1}{3a^3}.$$

Under different values of $a$, we have the following equilibria and payoffs.

- $\mathbf{1 < a \leqslant \frac{4}{3}}$: cNE $= (0, 1)$, $U_1 = \frac{1}{12}$, $U_0 = \frac{1}{6}$, $W = \frac{1}{2} + \frac{a}{2}$
- $\mathbf{\frac{4}{3} \leqslant a \leqslant \sqrt{3}}$: cNE $= (1, 1)$, $U_1 = \frac{1}{12} + \frac{1}{4a^2} - \frac{1}{3a^3}$, $U_0 = \frac{1}{2a^2}$, $W = \frac{1}{2a} + \frac{a}{2}$
- $\mathbf{a \geqslant \sqrt{3}}$: cNE $= (1, 0)$, $U_1 = \frac{1}{12}$, $U_0 = \frac{1}{6}$, $W = \frac{1}{2} + \frac{a}{2}$

Comparing with the result from the previous example (with $\alpha = 1/2$), we can see that noise reduction is profitable for workers and higher-ranked firms but completely disadvantageous for lower-ranked firms. Under incomplete information, $U_0$ has a chance to get types higher than just those below $\bar{x}$. In other words, a lower-ranked firm gets an advantage from errors in the evaluation process.

The last example shows the existence of Inclusive Equilibrium under the same conditions on the parameters except the distribution of workers.

**Example 4.** Let $\varepsilon \sim \mathbf{U}_{[-1/2,1/2]}$, $\bar{x} = 1/2$, $u_1 = \theta - 1/2$, $u_0 = \theta$, $F(\theta) = \theta^3$ ($\theta \in [0, 1]$). Instead of uniformly distributed workers in the previous example, now we have a distribution skewed towards higher types. Then

$$U_0^1 = 3\int_0^1 \theta^3(1-\theta)d\theta = \frac{3}{20}, \quad U_0^2 = 3\int_0^{1/a} \theta^3 d\theta = \frac{3}{4a^4},$$
$$U_1^1 = 3\int_0^1 (\theta - \frac{1}{2})\theta^3 d\theta = \frac{9}{40}, \quad U_1^2 = 3\int_{1/a}^1 (\theta - \frac{1}{2})\theta^3 d\theta = \frac{9}{40} + \frac{3}{8a^4} - \frac{3}{5a^5}.$$

Again, three cases exist.

- $\mathbf{1} < \mathbf{a} \leqslant \sqrt[4]{5}$: cNE $= (0, 1)$
- $\sqrt[4]{5} \leqslant \mathbf{a} \leqslant \mathbf{1.6}$: cNE $= (1/2, 1/2)$
- $\mathbf{a} \geqslant \mathbf{1.6}$: cNE $= (1, 0)$

In all cases, at least one of the firms is happy to get applications from the entire pool of workers. This happens because the quality of applicants is high. Thus, $U_1$ is willing to bear with a few low-skilled workers who occasionally get accepted, and $U_0$ receives a lot of profit from high-skilled workers who get rejected by $U_1$ "by mistake".

# 4 Case of three firms: $m = 2$

For simplicity, we consider only the most interesting case $p = \theta$ (and, therefore, $\bar{x} = 1/2$, $\varepsilon \sim \mathbf{U}_{[-1/2,1/2]}$) throughout the next two chapters.

This case includes competition among higher-ranked firms that changes the picture substantially (see Fig. 3). Also, under some values of costs workers may choose not only between the higher-ranked ($\theta \geqslant 1/2$) and the lower-ranked ($\theta < 1/2$) institutions, but between applying to either both risky firms and the safe one (for example, $c_1 = c_2 = 1/2, c_0 = 1$). In this case, $\hat{\theta}$ is different. The probability of being hired by at least one of the good firms is $(1-(1-p)^2)$, so each worker chooses between getting $a(1-(1-p)^2)$ and 1. Solving this, we find that if $p \geqslant 1 - \sqrt{1 - 1/a}$, a worker applies to $V_1$ and $V_2$, and otherwise she applies to $V_0$.

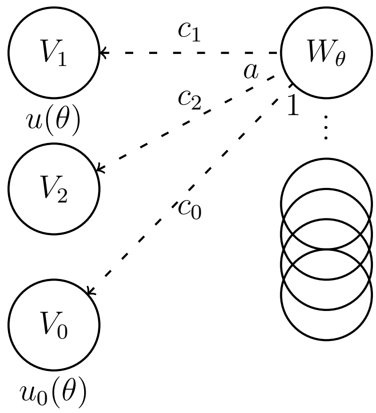


Figure 3: Case of three firms.

Proposition 2 in Appendix describes all the expected utilities of the firms for different actions. For each given $a$, $F(\theta)$, $u(\theta)$, and $u_0(\theta)$ we can find all best responses and thus all the possible pure strategy equilibria in the game. Consider several cases.

## 4.1 Positive Valuation of Workers by All Firms

Assume first that both higher-ranked firms welcome all types of workers.

**Example 5.** $a \geqslant 2$, $F(\theta) = \theta$, $u(\theta) = \theta$, $u_0(\theta) = \theta$. The only set of pure strategy Nash equilibria here satisfies the following condition:

$$c_1 + c_2 + c_0 \leqslant 1.$$

Firms' expected payoffs are $U = 0.2083$, $U_0 = 0.0833$. From the entire pool of workers, 33% go to $V_0$ in expectation and other 67% go to either $V_1$ or $V_2$. Each firm gets exactly 1/3 of the entire pool.

The cNE here is $(1/3, 1/3, 1/3)$. Thus, the costs form a negligible portion of the budget and go to zero when we increase number $m$ of higher-ranked firms. We call this *Inclusive Equilibrium.*

There exists only degenerate cNE $(0, 0, 1)$ if $a < 2$. Also, we can observe Inclusive Equilibrium under $a \geqslant 1.8002$ if the pool is skewed towards high-quality workers ($F(\theta) = \theta^2$) and under $a \geqslant 2.2497$ if the pool contains a lot of low-quality workers ($F(\theta) = 2\theta - \theta^2$). See the resulting Table 3.

| $F(\theta)$ | $(1/3, 1/3, 1/3)$ | $(0, 0, 1)$ |
|---|---|---|
| $\theta^2$ | $a \geqslant 1.8002$ | $a \leqslant 1.8002$ |
| $\theta$ | $a \geqslant 2$ | $a \leqslant 2$ |
| $2\theta - \theta^2$ | $a \geqslant 2.2497$ | $a \leqslant 2.2497$ |

Table 3: Equilibria with Positive Valuation for different $F$

## 4.2 Picky Firms

Now assume that the higher-ranked firms are not interested in bad workers.

**Example 6.** $a = 2$, $F(\theta) = \theta$, $u(\theta) = \theta - 1/2$, $u_0(\theta) = \theta$. We can see that $V_1$ and $V_2$ would like to avoid matching with all workers whose type is $\theta < 1/2$. There are two sets $C_1$ and $C_2$ of pure strategy Nash equilibria that correspondingly satisfy the following constraints:

$$\begin{cases} c_1 + c_2 + c_0 > 1, \\ c_1 + c_2 \leqslant 1, \\ c_1 + c_0 \leqslant 1, \\ c_2 + c_0 \leqslant 1, \\ c_1 \geqslant c_0, \ c_2 \geqslant c_0, \end{cases} \qquad \begin{cases} c_1 + c_2 > 1, \\ c_0 = 0. \end{cases}$$

Firms' expected payoffs are $U = 0.0495$, $U_0 = 0.0833$ for $C_1$ and $U = 0.0417$, $U_0 = 0.01667$ for $C_2$. From the entire pool of workers, 38% go to $V_0$ in expectation, 58% go to either $V_1$ or $V_2$, and 4% remain unmatched in the first case.

The cNE in the degenerate second case is $(1, 1, 0)$. For $C_1$, the cNE is $(1/2, 1/2, 1/2)$. We can see that costs here form a substantial part of the workers' budgets, although they leave some room for flexibility (workers can apply to both risky and safe firms). We call it *Selective Equilibrium*. The difference with Inclusive Equilibrium from Example 5 here is that costs do not converge to zero when $m \to \infty$.

**Example 7.** $a = 1.25$, $F(\theta) = \theta$, $u(\theta) = \theta - 1/2$, $u_0(\theta) = \theta$. Then the set of Nash equilibria can be defined by the following inequalities:

$$\begin{cases} c_1 + c_2 \leqslant 1, \\ c_1 + c_0 > 1, \;\; c_2 + c_0 > 1, \\ c_1 \leqslant c_0, \;\; c_2 \leqslant c_0. \end{cases}$$

Firms' expected payoffs are $U = 0.0593$, $U_0 = 0.1528$. From the entire pool of workers, 55% go to $V_0$ in expectation, 42% go to either $V_1$ or $V_2$, and 3% remain unmatched.

The cNE here is $(1/2, 1/2, 1)$. We know this situation as *Restrictive Equilibrium*. You can accept an offer from the safe firm and thus lose an opportunity to be considered by better firms, or you can reject it and have no safe option available. Such an aggressive policy allows $V_0$ to take 55% of the market. Notice that it works only if the difference between firms is small and, what is maybe even more important, workers' relative value of being hired is much higher.

Examining the entire range of $a$, we can see that there exists only degenerate cNE $(1, 1, 0)$ if $a > 2$ and $(0, 0, 1)$ if $a < 1.1655$. Two equilibria described in Example 6 may be observed for all $\sqrt{3} \leqslant a \leqslant 2$, and only one Selective Equilibrium is allowed when $1,2622 < a < \sqrt{3}$. Restrictive Equilibrium is supported by $1.1655 \leqslant a < 1.2622$. Another interesting situation may be observed under $a = 1.2622$.

**Example 8.** $a = 1.2622$, $F(\theta) = \theta$, $u(\theta) = \theta - 1/2$, $u_0(\theta) = \theta$. There are two sets $C_3$ and $C_4$ of pure strategy Nash equilibria that correspondingly satisfy the following constraints:

$$\begin{cases} c_1 + c_2 + c_0 > 1, \\ c_1 + c_2 \leqslant 1, \\ c_1 + c_0 \leqslant 1, \\ c_2 + c_0 \leqslant 1, \end{cases} \qquad \begin{cases} c_1 + c_2 \leqslant 1, \\ c_1 + c_0 > 1, \;\; c_2 + c_0 > 1, \\ c_1 \leqslant c_0, \;\; c_2 \leqslant c_0. \end{cases}$$

As before, $\mathrm{cNE}(C_3) = (1/2, 1/2, 1/2), \mathrm{cNE}(C_4) = (1/2, 1/2, 1)$. Firms' expected payoffs are $U = 0.0451$, $U_0 = 0.1481$ and $U = 0.0595$, $U_0 = 0.1481$, respectively. We can see that going down through $a = 1.2622$, $V_0$ jumps from Selective to Restrictive Equilibrium, delivering higher payoffs to everyone. Not taking into account degenerate equilibria, we can see that utility of $V_0$ monotonically increases while $a$ goes down from 2 to 1.1655, but utility of the higher-ranked firms goes in the opposite direction with a jump at $a = 1.2622$. For the full picture, see Table 4.

| $a$ | $(0, 0, 1)$ | $(1/2, 1/2, 1)$ | $(1/2, 1/2, 1/2)$ | $(1, 1, 0)$ |
|---|---|---|---|---|
| $1 < a < 1.1655$ | $+$ | $-$ | $-$ | $-$ |
| $1.1655 \leqslant a < 1.2622$ | $+$ | $+$ | $-$ | $-$ |
| $1.2622 \leqslant a < \sqrt{3}$ | $-$ | $+$ | $+$ | $-$ |
| $\sqrt{3} \leqslant a \leqslant 2$ | $-$ | $-$ | $+$ | $+$ |
| $2 < a$ | $-$ | $-$ | $-$ | $+$ |

Table 4: Equilibria for picky firms and different $a$ ($F(\theta) = \theta$)

## 4.3 Picky Firms and Low-Quality Workers

**Example 9.** $a = 2$, $F(\theta) = 2\theta - \theta^2$, $u(\theta) = \theta - 1/2$, $v_0(\theta) = \theta$. There are two different sets $C_5$ and $C_6$ of pure strategy Nash equilibria here:

$$\begin{cases} c_1 + c_2 + c_0 > 1, \\ c_1 + c_2 \leqslant 1, \ c_1 + c_0 \leqslant 1, \ c_2 + c_0 \leqslant 1, \\ c_1 \geqslant c_0, \ c_2 \geqslant c_0 \end{cases} \qquad \text{and} \qquad \begin{cases} c_1 + c_2 > 1, \\ c_1 + c_0 > 1, \\ c_2 + c_0 > 1. \end{cases}$$

Firms' expected payoffs a) in the first case: $U = 0.0036$, $U_0 = 0.1146$; b) in the second case: $U = 0.0156$, $U_0 = 0.1667$. Thus, the second equilibrium is Pareto optimal for the firms. From the entire pool of workers 58% go to $V_0$ in expectation, 39% go to either $V_1$ or $V_2$, and 3% remain unmatched in the first case, and 75% go to $U_0$ in expectation, 17% go to either $V_1$ or $V_2$, and 8% remain unmatched in the second case.

We can see that $\mathrm{cNE}(C_5) = (1/2, 1/2, 1/2)$ and $\mathrm{cNE}(C_6) = (1, 1, 1)$. In the second equilibrium, firms use the hardest possible screening to force low-quality workers apply only to $V_0$. Thus, each worker is allowed to submit only one application. This is a typical market of scientific journals and authors (if we consider it decentralized). There are good journals where a paper can easily be rejected and mediocre ones where a paper will most likely be published. Depending on the quality $\theta$ of the article, the author must make a decision where to submit. The major cost in this case is probably time, not money, and it is more or less the same for any journal. It might have been allowed for papers to be submitted to two different journals (the first equilibrium), but this is not a Pareto optimal strategy for journals.

**Example 10.** $a = 1.25$, $F(\theta) = 2\theta - \theta^2$, $u(\theta) = \theta - 1/2$, $u_0(\theta) = \theta$. There are two different sets $C_7$ and $C_8$ of pure strategy Nash equilibria again:

$$\begin{cases} c_1 + c_0 > 1, \ c_2 + c_0 > 1, \\ c_1 \geqslant c_0, \ c_2 \geqslant c_0 \end{cases} \qquad \text{and} \qquad \begin{cases} c_1 + c_2 \leqslant 1, \\ c_1 + c_0 > 1, \ c_2 + c_0 > 1, \\ c_1 \leqslant c_0, \ c_2 \leqslant c_0. \end{cases}$$

Firms' expected payoffs a) in the first case: $U = 0.0064$, $U_0 = 0.2987$; b) in the second case: $U = 0.0188$, $U_0 = 0.193$. From the entire pool of workers 96% go to $V_0$ in expectation, 3% go to either $V_1$ or $V_2$, and 1% remain unmatched in the first case, and 80% go to $V_0$ in expectation, 18% go to either $V_1$ or $V_2$, and 2% remain unmatched in the second case.

Here, cNE$(C_7) = (1, 1, 1)$ and cNE$(C_8) = (1/2, 1/2, 1)$. To choose a more appropriate equilibrium, fix $c_0 = 1$ and focus on the behavior of $V_1$ and $V_2$. Their part of the game may be characterized by the next payoffs table:

| $V_1 \setminus V_2$ | $1/2 : c_1 + c_2 \leqslant 1$ | $1 : c_1 + c_2 > 1$ |
|---|---|---|
| $1/2 : c_1 + c_2 \leqslant 1$ | 0.0188 | 0.0064 |
| $1 : c_1 + c_2 > 1$ | 0.0064 | 0.0064 |

We can see that strategies 1 are weekly dominated by strategies 1/2 for both firms. Restrictive cNE $(1/2, 1/2, 1)$ looks more reasonable here.

Note that pools with high-quality workers (for example, $F(\theta) = \theta^2$) do not give us any information because no pure strategy Nash equilibria exist there. It is not the case in the next subsection though.

We can describe now how our model explains the differences in application costs on American and European Econ PhD markets by different evaluation of students. Prestigious universities in the US are not interested in enrolling low-quality students whose possible struggling with the program and bad placement may worsen the reputation of a school and decrease their teaching standards. On the other hand, universities in Europe are less competitive and more socially oriented. More positive evaluation of students in Europe and existence of Inclusive (European) equilibrium even with skewed to the right densities may also imply that prospective PhD students' pools in Europe are of higher quality on average. Of course, this is just a speculation, since we do not know all the mechanisms behind costs setting. For example, it may be the case that in some countries in Europe governments require implementing zero costs.

# 5 General Case of $m+1$ Firms

Consider necessary and sufficient conditions for some of equilibria from the previous chapter in the general case.

**Proposition 3.** Inclusive Equilibrium with $C' = \{c : c_1 + c_2 + \cdots + c_m + c_0 \leqslant 1\}$ and $\mathrm{cNE}(C') = (\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}, \frac{1}{m+1})$ exists if and only if

$$\int_0^1 u_0(p)(1-p)^m dF(p) \geqslant \int_0^{1-\sqrt[t]{1-\frac{1}{a}}} u_0(p)(1-p)^{m-t} dF(p) \quad (t = 1, 2, \ldots, m),$$
$$\int_0^{1/a} u(p)(1-p)^{m-1} p dF(p) \geqslant 0,$$
$$\int_0^{1/a} u(p)(1-(1-p)^m) dF(p) \geqslant 0.$$

**Proposition 4.** Selective Equilibrium with $C'' = \{c : c_i + c_j + c_0 > 1, c_i + c_j \leqslant 1, c_i + c_0 \leqslant 1, c_0 \leqslant c_i (i, j \in \{1, \ldots, m\})\}$ and $\mathrm{cNE}(C'') = (1/2, 1/2, \ldots, 1/2, 1/2)$ in symmetric case with maximum costs exists if and only if

$$\int_0^{1/a} u_0(p)(1-p) dF(p) \geqslant \int_0^1 u_0(p)(1-p)^2 dF(p),$$
$$\int_0^{1/a} u_0(p)(1-p) dF(p) \geqslant \int_0^{1-\sqrt{1-\frac{1}{a}}} u_0(p) dF(p),$$
$$\frac{2}{m} \int_{1/a}^1 u(p) \frac{1-(1-p)^2}{2} dF(p) + \frac{1}{m} \int_0^{1/a} u(p) p dF(p) \geqslant$$
$$\geqslant \max\{\int_0^1 u(p) \frac{1-(1-p)^2}{2} dF(p), 0\},$$
$$\int_{1/a}^1 u(p) \frac{1-(1-p)^2}{2} dF(p) \geqslant 0.$$

**Proposition 5.** Restrictive Equilibrium with $C''' = \{c : c_1 + c_2 + \ldots + c_m \leqslant 1, c_i + c_0 > 1, c_i \leqslant c_0 (i \in \{1, \ldots, m\})\}$ and $\mathrm{cNE}(C''') = (1/m, 1/m, \ldots, 1/m, 1)$

in symmetric case with maximum costs exists if and only if

$$\int_0^{1-\sqrt[m]{1-\frac{1}{a}}} u_0(p)dF(p) \geqslant \int_0^{1-\sqrt[t]{1-\frac{1}{a}}} u_0(p)(1-p)^{m-t}dF(p),$$
$$(t = 0, 1, \ldots, m-1),$$

$$\int_{1-\sqrt[m]{1-\frac{1}{a}}}^1 u(p)\frac{1-(1-p)^m}{m}dF(p) \geqslant 0,$$

$$\int_{1-\sqrt[m]{1-\frac{1}{a}}}^1 u(p)\frac{1-(1-p)^m}{m}dF(p) \geqslant$$
$$\geqslant \frac{m-1}{m}\int_{1-\sqrt[m]{1-\frac{1}{a}}}^1 u(p)\frac{1-(1-p)^{m-1}}{m-1}dF(p),$$

$$\int_{1-\sqrt[m]{1-\frac{1}{a}}}^1 u(p)\frac{1-(1-p)^m}{m}dF(p) \geqslant$$
$$\geqslant \int_{1-\sqrt[m-1]{1-\frac{1}{a}}}^1 u(p)\frac{1-(1-p)^m}{m}dF(p) + \int_0^{1-\sqrt[m-1]{1-\frac{1}{a}}} u(p)pdF(p).$$

# 6 Conclusion

We consider a model of two-sided market where agents on one side (firms) set their admission thresholds $\bar{x}$ and/or application costs $c$, and agents on the other side (workers) apply restricted by their limited budget. These thresholds and costs serve as a screening instrument that allows higher-ranked firms to regulate number and quality of workers. We find the optimal value of $\bar{x}$ in the simplest case of two firms for some values of parameters.

The special case of our model reasonably describes a market of prospective PhD students giving possible insights concerning existing equilibria in North American and European schools. We also get new information about some other equilibria that appear in our model, e.g. restrictive equilibria or single applications. In particular, we show that Restrictive Equilibrium is beneficial not only for a weaker firm but also for a better institution: the latter deals with a higher-quality pool of candidates. Inclusive (European) equilibrium may be driven by more positive evaluation of workers (students) and less competitive environment.

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# Appendix

**Proposition 1.** Let

$$U_0^1 = \int_0^1 u_0(\theta) G(\bar{x} - \theta) dF(\theta), \qquad U_0^2 = \int_0^{\hat{\theta}} u_0(\theta) dF(\theta),$$
$$U_1^1 = \int_0^1 u_1(\theta)(1 - G(\bar{x} - \theta)) dF(\theta), \quad U_1^2 = \int_{\hat{\theta}}^1 u_1(\theta)(1 - G(\bar{x} - \theta)) dF(\theta).$$

Then the game on Fig. 2 has the following set of (subgame perfect) Nash equilibria ($0 \leqslant c_1 \leqslant 1$, $0 \leqslant c_0 \leqslant 1$ if not stated otherwise):

1. if $U_0^1 > U_0^2$, $U_1^1 > U_1^2$, $U_1^1 \geqslant 0$, then the NE set is $(c_1, c_0)$, $c_1 + c_0 \leqslant 1$ with cNE $(1/2, 1/2)$
2. if $U_0^1 > U_0^2$, $U_1^1 < U_1^2$, $U_1^1 \geqslant 0$, then the NE set is $(c_1, 0)$ with cNE $(1, 0)$
3. if $U_0^1 > U_0^2$, $U_1^1 < 0$, $U_1^2 \geqslant 0$, then the only NE is $(1 + \delta, 0)$
4. if $U_0^1 < U_0^2$, $U_1^1 < U_1^2$, $U_1^2 \geqslant 0$, then the NE set is $(c_1, c_0)$, $c_1 + c_0 > 1$ with cNE $(1, 1)$
5. if $U_0^1 < U_0^2$, $U_1^1 > U_1^2$, $U_1^1 \geqslant 0$, then the NE set is $(0, c_0)$ with cNE $(0, 1)$
6. if $U_1^1 < 0$, $U_1^2 < 0$, then the NE set is $(1 + \delta, c_0)$ with cNE $(1 + \delta, 1)$.

**Proof.** Consider all four integrals. $U_0^1$ and $U_1^1$ are aggregate utilities of $V_0$ and $V_1$, respectively, when $c_0 + c_1 \leqslant 1$. Indeed, assume any worker can apply to both firms. Then the integral bounds are $[0, 1]$. The higher-ranked firm hires the worker of type $\theta$ with probability $1 - G(\bar{x} - \theta)$ and gets utility $u_1(\theta)$. The safe firm can hire the worker of type $\theta$ (and get utility $u_0(\theta)$) only if she was rejected by $V_1$. The probability of this event is $G(\bar{x} - \theta)$.

Similarly, $U_0^2$ and $U_1^2$ are aggregate utilities of $V_0$ and $V_1$, respectively, when $c_0 + c_1 > 1$, $c_0 \leqslant 1$, $c_1 \leqslant 1$. Now workers must choose between two firms. The high quality workers ($\theta > \hat{\theta}$) choose the higher-ranked firm, and the low quality workers ($\theta < \hat{\theta}$) choose the safe firm. Since workers now apply only to one firm, everyone who applies to $V_0$ will be hired for sure. However, the probability of being enrolled to $V_1$ is still $1 - G(\bar{x} - \theta)$.

First three cases describe the willingness of $V_0$ to share the market. In case 1, $V_1$ also wants to share the market. Thus, they both agree on keeping the total costs below or equal to the worker's budget. In case 2 though, $V_1$ would like to split the market and hire only the high quality workers. It means that for any $c_0 > 0$, $V_1$ would be able set $c_1 \leqslant 1$ such that $c_1 + c_0 > 1$. The only option for $V_0$ to avoid that situation is to set $c_0 = 0$. Then $V_1$ has to choose between getting $U_1^1$ or leaving the market with $U_1 = 0$. Since $U_1^1 \geqslant 0$, the only equilibrium here is $(c_1, 0)$. Finally, case 3 is similar to case 2, but here it is not profitable for $V_1$ to enter the market ($U_1^1 < 0$). Thus, $c_1 = 1 + \delta$, and $V_0$ gets the entire pool of workers.

Cases 4–5 describe the willingness of $V_0$ to split the market. Again, if $V_1$ also wants to split, there is no problem with that: case 4 works with equilibrium $c_1 + c_0 > 1$. However, $V_1$ may want to share the market (case 5). Then we observe the behavior symmetric to case 2.

Finally, the degenerate case 6 describe the situation when $V_1$ does not enter the market under any conditions. In this case, $V_0$ does not even have to keep zero costs: any level of participation $c_0 \leqslant 1$ would be fine. ■

**Proposition 2.** The overall expected utilities of all the firms in the game on Fig. 3 are the following.

**I** Utility for $V_2$ (we can get the same for $V_1$ simply by interchanging costs' subindices $1 \leftrightarrow 2$)

- If $c_1 + c_0 \leqslant 1$ and $c_1 \geqslant c_0$ then

$$U = \begin{cases} \int_0^1 u(p)\frac{1-(1-p)^2}{2}dF(p) & c_2 + c_1 + c_0 \leqslant 1, \\ \int_{\frac{1}{a}}^1 u(p)\frac{1-(1-p)^2}{2}dF(p) + \frac{1}{2}\int_0^{\frac{1}{a}} u(p)pdF(p) & \sum c_i > 1, c_2 + c_1 \leqslant 1, \\ \frac{1}{2}\int_0^1 u(p)pdF(p) & c_2 + c_1 > 1, c_2 + c_3 \leqslant 1, \\ 0 & c_2 + c_3 > 1 \end{cases}$$

- If $c_1 + c_0 \leqslant 1$ and $c_1 < c_0$ then

$$U = \begin{cases} \int_0^1 u(p)\frac{1-(1-p)^2}{2}dF(p) & c_2 + c_1 + c_0 \leqslant 1, \\ \int_{\frac{1}{a}}^1 u(p)\frac{1-(1-p)^2}{2}dF(p) + \frac{1}{2}\int_0^{\frac{1}{a}} u(p)pdF(p) & \sum c_i > 1, c_2 + c_0 \leqslant 1, \\ \int_{\frac{1}{a}}^1 u(p)\frac{1-(1-p)^2}{2}dF(p) & c_2 + c_0 > 1, c_2 + c_1 \leqslant 1, \\ 0 & c_2 + c_1 > 1 \end{cases}$$

- If $c_1 + c_0 > 1$ and $c_1 \geqslant c_0$ then

$$U = \begin{cases} \int_{\frac{1}{a}}^1 u(p)\frac{1-(1-p)^2}{2}dF(p) + \int_0^{\frac{1}{a}} u(p)pdF(p) & c_2 + c_1 \leqslant 1, \\ \int_0^1 u(p)pdF(p) & c_2 + c_1 > 1, c_2 + c_3 \leqslant 1, \\ \frac{1}{2}\int_{\frac{1}{a}}^1 u(p)pdF(p) & c_2 + c_3 > 1, c_2 \leqslant 1, \\ 0 & c_2 > 1 \end{cases}$$

- If $c_1 + c_0 > 1$ and $c_1 < c_0$ then

$$U = \begin{cases} \int_{\frac{1}{a}}^{1} u(p)\frac{1-(1-p)^2}{2}dF(p) + \int_0^{\frac{1}{a}} u(p)pdF(p) & c_2 + c_0 \leqslant 1, \\ \int_{1-\sqrt{1-\frac{1}{a}}}^{1} u(p)\frac{1-(1-p)^2}{2}dF(p) & c_2 + c_0 > 1, c_2 + c_1 \leqslant 1, \\ \frac{1}{2}\int_{\frac{1}{a}}^{1} u(p)pdF(p) & c_2 + c_1 > 1, c_2 \leqslant 1, \\ 0 & c_2 > 1. \end{cases}$$

**II** Utility for $V_0$

- If $c_1 + c_2 \leqslant 1$ then

$$U_0 = \begin{cases} \int_0^1 u_0(p)(1-p)^2 dF(p) & c_0 + c_1 + c_2 \leqslant 1, \\ \int_0^{\frac{1}{a}} u_0(p)(1-p)dF(p) & c_0 + c_1 + c_2 > 1, c_0 + \min\{c_1, c_2\} \leqslant 1, \\ \int_0^{1-\sqrt{1-\frac{1}{a}}} u_0(p)dF(p) & c_0 + \min\{c_1, c_2\} > 1, c_0 \leqslant 1, \\ 0 & c_0 > 1 \end{cases}$$

- If $c_1 + c_2 > 1$ and $\min\{c_1, c_2\} \leqslant 1$ then

$$U_0 = \begin{cases} \int_0^1 u_0(p)(1-p)dF(p) & c_0 + \min\{c_1, c_2\} \leqslant 1, \\ \int_0^{\frac{1}{a}} u_0(p)dF(p) & c_0 + \min\{c_1, c_2\} > 1, c_0 \leqslant 1, \\ 0 & c_0 > 1 \end{cases}$$

- If $\min\{c_1, c_2\} > 1$ then

$$U_0 = \begin{cases} \int_0^1 u_0(p)dF(p) & c_0 \leqslant 1, \\ 0 & c_0 > 1. \end{cases}$$