

# Near-optimal Storage Strategies in Electricity Markets

Sergei Balakin\* and Guillaume Roger\*

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## Abstract

Storing electricity is completely essential to the energy transition. It also deeply disrupts the manner in which electricity markets operate, for it introduces delay. In this paper, we consider a dynamic model of an oligopolistic market with demand shocks, in which a storage unit buys and sells energy subject to a capacity constraint. To make progress in this stochastic game with constraints, we restrict attention to simple heuristics, and we can characterise the optimal policy of a storage unit in this restricted class of heuristics. The heuristics, the exogenous stochastic process and the capacity constraint interact to induce rich dynamics. The optimal policy is sensitive to the nature of demand shocks and to storage capacity. For a fixed capacity, the storage unit internalises its unilateral market power; it acts like a monopolist on its arbitrage spread. The optimal capacity is also interior because of uncertainty: it is costly to be stuck full or empty, and that cost becomes overwhelming as capacity increases. We construct an equilibrium, in which electricity arbitrage is never profitable, and so conclude that successful entry is not a foregone conclusion.

This work informs market participants as well as the design of electricity markets with storage. It is particularly relevant to major markets with rapid penetration of renewable energy sources, like California or Australia. It can also be applied to trading securities.

Key words: *stochastic game, dynamic trading, energy storage*

JEL: *C73, D43, D47, Q41, Q42*

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\*Monash University. Corresponding author: Roger – guillaume.roger72@gmail.com. Funding from ARENA under grant 2021/ARP016 gratefully acknowledged.

# 1 Introduction

Producing electricity on a large scale and securely so as to transition away from polluting fossil fuels requires vast amounts of storage. That storage delivers the intertemporal smoothing of consumption and production that is completely essential to effecting the energy transition. In the State of South Australia, for example, there is no object in installing more renewable generation capacity as the State can power itself completely from renewable energy when it is available.<sup>1</sup> However there is a pressing need for storage. As of June 2023, California features 5 GW of storage capacity; to reach the legally-mandated 90% zero-emission energy target by 2030, it needs approximately 65 GWh of energy capacity – approximately 6.5 times the current level.<sup>2</sup> This represents an enormous investment. Yet we know very little of the economics of electricity storage, because for long it was simply not an option.

This paper addresses this gap – in part. We study a model of electricity trading based on storage over a long horizon and rooted in an oligopolistic market. Conventional generators produce for immediate sale, and demand is subject to aggregate shocks.<sup>3</sup> A storage operator can step in and implement the simple idea of “buying low and selling high”, the details of which are in fact quite complicated and rich.

Our main contribution is this. A storage operator must balance two essential forces: its unilateral market power (current quantity) and continuation (the value of future trades). The novelty lies in how this well-known trade-off materialises. On the first account, a storage unit with large enough a capacity internalises its own market power and thus withholds quantities. On the second one, it must have enough capacity to fully exploit the intertemporal energy arbitrage, but not too much so as to not face large costs if being stuck empty and unable to charge, or worse, full and unable to sell, which happens with positive probability. We call this the *continuation risk*.<sup>4</sup> Its cost increases in capacity, whence the trade-off. A storage

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<sup>1</sup>Source: RenewEconomy. <https://reneweconomy.com.au/south-australias-remarkable-100-per-cent-renewables-run-extends-to-over-10-days/>.

<sup>2</sup>Source: CEC report on energy.ca.gov: <https://www.energy.ca.gov/news/2023-05/new-data-shows-growth-californias-clean-electricity-portfolio-and-battery>. The data shows 5GW, that is, power rating; assuming a (generous) 2-hour duration, this gives 10 GWh, which is energy rating.

<sup>3</sup>One can also add renewable energy with stochastic supply and conceive of the demand function as residual demand without material consequences.

<sup>4</sup>This effect is asymmetric because of efficiency losses; it is more costly to be stuck full than empty.

operator can delay this event by reducing the fraction of capacity traded. But this risk never disappears, except (asymptotically) when shocks are governed by a low-persistence Markov chain; then the continuation risk is negated.<sup>5</sup> Capacity is thus also optimally interior. This is really new, and completely moot in the absence of market power. Indeed, if capacity is small enough, the cost of the continuation risk is trumped by the arbitrage gain; there is no withholding. We explore some extensions, including the role of capacity constraints; all confirm our results.<sup>6</sup>

Our positive analysis suggests some tentative implications for competition policy and market design. The market is better served by a large fleet of small units; a market operator may improve outcomes by offering a measure of insurance. An open question remains: how to best determine the bidding space and clear a market when bidders play dynamic strategies? We also show the success of a nascent storage industry cannot be taken for granted. Indeed we construct a collusive equilibrium, in which generators are able to act tacitly to prevent a storage unit from operating.<sup>7</sup> It shows that some measure of support may be necessary to a successful entry.

Our second contribution is more technical; we claim some progress in characterising behaviour (not quite equilibrium) in a dynamic game. In the stochastic game ([Shapley \(1953\)](#)) we study, which admits a very large number of equilibria, we must limit ourselves to studying heuristics rather than the more desirable equilibrium strategies that remain out of reach. More precisely we must fix a behaviour of the conventional generators and select a heuristic of the storage operator. With this restriction, we can characterise optimal heuristics in their class, and study some comparative statics. For simple heuristics, we uncover a recursive structure that is tractable and allows us to compute the corresponding value function, which can then be optimised. While these heuristics commit the storage operator to a fixed behaviour over time, this property is not essential to our results in a sense we make precise. As in any stochas-

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<sup>5</sup>To be sure this is not a quantity effect: these traded quantities are almost constant in capacity. It is the cost of a long string of shocks in the same direction.

<sup>6</sup>Introducing a capacity constraint has the same effect as a higher marginal cost, so it speaks to heterogeneity as well.

<sup>7</sup>More precisely, they are able to render arbitrage unprofitable if the storage operator ever decides to incur the charging cost. This equilibrium is subtle and interesting in its own right.

tic game, the strategies interact with the exogenous stochastic process to induce endogenous transitions between states. Here, this endogenous stochastic process is further enriched by *constraints* on capacity and on the initial condition to generate novel dynamics. To the extent that capacity constraints are the norm rather than the exception, this is an important contribution. This approach allows us to extend the independent case to a richer, Markovian shock structure with more or less persistence in shocks.

The model we present in this paper can be used, possibly with some adaptations, to market making in securities. Indeed, this intermediation activity shares many characteristics with trading electricity through storage: assets are bought and sold, a revenue is generated by arbitrage, holding inventory is necessary and price impact matters a great deal. The works of [Vayanos \(1999\)](#) and [Glebkin et al. \(Forthcoming\)](#) differ from ours in that traders seek to diversify idiosyncratic risk, whereas here the arbitrageur takes advantage of aggregate risk, and so *supplies* insurance.

Storage has been in existence for some time in the form of hydro-electric power. However storing water to generate electricity differs from having to first *purchase* electricity in order to sell it later. Once a dam is built, the water inflow is free, exogenous and stochastic; in contrast, a storage unit pays for the energy it buys, it can have (a measure of) monopsony power, and it makes that decision optimally as part of its trading strategy. Furthermore, most models of dam management amount to an optimal control problem rather than a game, and ignore completely the market power of the dam operator on prices. We show that both market power and having to buy energy are first-order considerations for a storage operator. This work is also conceptually connected to the inventory management problem; see [Harrison and Taylor \(1978\)](#) for example. However that problem is strictly one of stochastic control – not a game, in which the per-unit payoffs (rewards or costs) are exogenous. In our model, prices are determined endogenously, which gives rise to the trade-offs we mentioned earlier.

This paper is one of very few on the economics of electricity storage. It distinguishes itself from the extant literature because it seeks to characterise behaviour in a stochastic environment with market power. [Karaduman \(2020\)](#) is the first to study grid scale storage,

using Australian data from the National Electricity Market. Generators and the storage unit play an infinite horizon game and market power is internalised. However, [Karaduman \(2020\)](#) does not compute the best reply; rather he simulates it from the data. Hence the actual behaviour of the storage unit is never known. [Andres-Cerezo and Fabra \(2023\)](#) study the question of market structure with storage, but leave aside how storage actually behaves. They ask in particular “who should (or not) own storage units”. A generator enhances its market power by also owning storage, especially at time when demand is the highest: in those times the substitutability between storage and generation should be exploited to its fullest, but the joint ownership of these two assets induces more quantity withholding. [Butters et al. \(Working Paper\)](#) use California data to estimate the equilibrium effect of large-scale storage. As the storage fleet expands, arbitrage revenue decreases, which hinders adoption. In that model however storage is assumed to behave competitively. We study the details of buying and selling with market power. [Schmalensee \(2022\)](#) studies storage investment, which we take as exogenous; he models the intra-day behaviour of storage rather than short-term arbitrage opportunities. Energy generation and storage are competitive rather than oligopolistic.<sup>8</sup> [Williams and Green \(2022\)](#) compute the welfare effects of storage on the current British market using simulations, and so without characterising any equilibrium, with time-varying demand and no uncertainty. [Geske and Green \(2020\)](#) do study arbitrage in a model of imperfect competition with demand uncertainty and diurnal, weekly and seasonal patterns. In such a complicated environment they must limit themselves to numerical (approximate) solutions to the welfare maximization problem. They also exhibit precautionary behaviour – in our case, quantity withholding. We show that market power and uncertainty are critical aspects of the problem.

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<sup>8</sup>[Schmalensee \(2022\)](#) also assumes that storage is fully discharged after the “nighttime”, while we let the storage operator make that decision in equilibrium.

## 2 Model

Consider a market with one storage unit,  $n$  electricity generators labelled  $j = 1, 2, \dots, n$ , and a pool of consumers. We simplify institutional details so that retailers and consumers are con-founded and retailing has no cost; another way of saying this is that retailers perfectly reflect the behavior of consumers. That behavior is described by the demand function  $D(p_t, \varepsilon_t)$  for each period  $t$ , where  $\varepsilon_t$  is a shock distributed according to some commonly known distribution  $F$ . Each of the generators  $j$  produces a quantity of energy  $q_t^j$  for each period  $t$ , and may or may not be subject to capacity constraints. The storage unit has finite capacity  $k$ . In each period, it can either buy energy (charge) up to its capacity, or sell any amount of available energy (discharge).<sup>9</sup> This process can be described formally by a simple equation of motion:

$$c_t = c_{t-1} + b_t - \frac{s_t}{\delta}, \quad t \in \mathbb{N}, \quad c_0 = 0. \quad (1)$$

Here,  $c_t$  is a current level of charge ( $0 \leq c_t \leq k$ ),  $\delta$  is a round-trip efficiency parameter ( $0 < \delta \leq 1$ ), and  $b_t \geq 0, s_t \geq 0$ . A storage operator can only either buy or sell in each period, so  $b_t \cdot s_t = 0$  for any  $t$  – this is a technical characteristic. The market clears if

$$D(p_t, \varepsilon_t) = \sum_{j=1}^n q_t^j - b_t + s_t$$

for any  $t$ , where we suppose that players engage in Cournot competition, which requires some justification. The norm in electricity markets is to use the more elegant supply-function equilibrium (SFE); however the richness of the SFE is lost here since we rely throughout on binary shocks; see [Klemperer and Meyer \(1989\)](#). Further, the Cournot outcome is a possible equilibrium outcome of the SFE and constitutes an upper bound for the payoffs to suppliers ([Klemperer and Meyer \(1989\)](#)).<sup>10</sup> Finally, Cournot competition is used as a successful proxy

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<sup>9</sup>We make no distinction between power and energy; it is as if a quantity were either energy or power for a prescribed duration (e.g. for the trading interval).

<sup>10</sup>[Vayanos \(1999\)](#) and [Glebkin et al. \(Forthcoming\)](#) do use the SFE with a normal distribution of shocks with full support and exponential utility, whence supply bids are linear. While the equilibrium is linear when demand is linear in a one-shot game ([Klemperer and Meyer \(1989\)](#)), it is not clear that it must be so in our dynamic game.

in many papers ([Acemoglu et al. \(2017\)](#), [Willems et al. \(2009\)](#), [Lundin and Tangerås \(2020\)](#)); much of this work relies on the estimations of [Borenstein and Bushnell \(1999\)](#), [Borenstein et al. \(1999\)](#) or [Bushnell et al. \(2008\)](#). Since the nature of competition is not the primary object of interest, throughout the rest of the paper we consider a linear demand function:

$$D(p_t, \varepsilon_t) = 1 - p_t + \varepsilon_t.$$

Rather, the goal is to find optimal policies  $\{b_t, s_t\}_{t=0}^{\infty}$  which are the part of a dynamic Nash equilibrium. We suppose the storage unit has a discount factor  $\beta < 1$ ; it is exposed to a strictly positive interest rate. Depending on the decisions of the storage operator, in each round there may be either

- $n$  (symmetric) competitors; or
- $n + 1$  competitors, with the storage unit having a limited capacity.

In an extension we let a subset of the generators be capacity constrained, which has the same effect as introducing heterogeneous technologies. The results are those we expect. First we characterise the optimal variables of a static problem, which are useful throughout.<sup>11</sup>

**Lemma 1.** *If the storage unit is a seller with  $c$  units of energy available, then the (symmetric) equilibrium price  $p^*$  and equilibrium quantities  $s^*$  and  $q^*$  under Cournot competition are:*

$$p^* = \frac{1 + \varepsilon - c}{n + 1}, \quad s^* = c, \quad q^* = \frac{1 + \varepsilon - c}{n + 1} \quad \text{if } c \leq \frac{1 + \varepsilon}{n + 2}; \quad (2)$$

$$p^* = \frac{1 + \varepsilon}{n + 2}, \quad s^* = q^* = \frac{1 + \varepsilon}{n + 2} \quad \text{if } c > \frac{1 + \varepsilon}{n + 2}. \quad (3)$$

**Proof.** In the first case, we have Cournot competition between  $n + 1$  players where one of the players has limited capacity. Standard Cournot competition between  $n + 1$  players is observed in the second case. ■

**Lemma 2.** *If the storage unit is a buyer with willingness to purchase  $c$  units, then the (symmetric) equilibrium price  $p^*$  and equilibrium quantities  $b^*$  and  $q^*$  under Cournot competition*

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<sup>11</sup>We omit the index  $t$  if it does not lead to confusion.

are

$$p^* = \frac{1 + \varepsilon + c}{n + 1}, \quad b^* = c, \quad q^* = \frac{1 + \varepsilon + c}{n + 1}. \quad (4)$$

The proof is trivial and therefore omitted. If the storage unit neither buys nor sells, standard Cournot competition between  $n$  generators prevails, and then  $p^* = q^* = (1 + \varepsilon)/(n + 1)$  in the symmetric equilibrium. Next we turn to the object of this paper, which is trading in the dynamic game. We begin with a special case as an example.

### 3 An introductory example

Let shocks  $\varepsilon_t$  be independently and identically distributed,

$$Pr\{\varepsilon = a\} = Pr\{\varepsilon = -a\} = 1/2, \quad 0 < a < 1 \quad (5)$$

for any  $t$ . Suppose also that the storage operator can only either charge or discharge in full; it can buy either 0 or  $k$  units of energy, or sell either 0 or  $\delta k$  units at each period of time. (Thus, option (3) is not available.) For convenience, we define charging costs (when purchasing energy) under the negative shock as  $B$  and likewise the revenue it earns when selling energy under the positive shock as  $A$ :

$$B = B(k) = \frac{1 - a + k}{n + 1} \cdot k, \quad A = A(k) = \frac{1 + a - \delta k}{n + 1} \cdot \delta k.$$

Observe that it cannot be optimal to charge when the shock  $\varepsilon$  is positive, nor can it be optimal to discharge when it is negative. Let also the coefficients

$$G_{01} = \left(\frac{1 + a}{n + 1}\right)^2, \quad G_{00} = \left(\frac{1 - a + k}{n + 1}\right)^2, \quad G_{10} = \left(\frac{1 - a}{n + 1}\right)^2, \quad G_{11} = \left(\frac{1 + a - \delta k}{n + 1}\right)^2.$$

$G_{ij}$  is a (non-discounted) generator's payoff when the storage is either empty ( $i = 0$ ) or full ( $i = 1$ ) and when the shock is either negative ( $j = 0$ ) or positive ( $j = 1$ ). Suppose also the



discount factor  $\beta$  is such that

$$B < \frac{\beta}{2 - \beta} A. \textcolor{blue}{12}$$

Then a dynamic equilibrium exists and is characterised as follows:<sup>[13](#)</sup>

- the empty storage buys  $k$  units with the first negative shock and sells  $\delta k$  units with the first positive shock afterwards;
- in each period, the generators set quantities  $q^*$  according to static Cournot competition and based on the current shock and the state of storage (full or empty). Namely,

$$\begin{aligned} \text{when storage is empty:} \quad q^* &= \frac{1+a}{n+1} \text{ if } \varepsilon = a, & q^* &= \frac{1-a+k}{n+1} \text{ if } \varepsilon = -a; \\ \text{when storage is full:} \quad q^* &= \frac{1-a}{n+1} \text{ if } \varepsilon = -a, & q^* &= \frac{1+a-\delta k}{n+1} \text{ if } \varepsilon = a. \end{aligned}$$

Cumulative consumers' expected payments per period  $C_1$  are

$$C_1 = \frac{1}{2(n+1)^2} (2n(1+a^2) - ka(n-1) - k^2).$$

Expected payoffs  $U_s$  and  $U_g$  of the storage unit and the generators, respectively, take the following form:

$$\begin{aligned} U_s &= \frac{1}{2} \left[ -B + \frac{\beta}{2(1-\beta)} (A - B) \right], \\ U_g &= \frac{1}{2} (G_{01} + G_{00}) + \frac{\beta}{4(1-\beta)} (G_{10} + G_{11} + G_{00} + G_{01}). \end{aligned} \tag{6}$$

Of course, this equilibrium is not the only one, but it is one that can be used to explore some of, but not all, the salient features of the general problem. First, we see that storage increases the output of generators when the demand shock is negative; this also increases prices (when they are otherwise low). Storage also decreases the output of these generators when the shock is positive; this concurrently depresses otherwise high prices. Hence every time it engages in

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<sup>12</sup>Note  $A$  and  $B$  are determined in terms of primitives. This is just convenient notation.

<sup>13</sup>We prove this claim formally in the Appendix.

arbitrage, storage decreases the very spread it feeds on.

Second, in the first instance, storage is a *complement* to generators but in the second one, is a *substitute* for generators. These patterns are more than astute observations if considering entry of new generation capacity; for solar generation, for example, storage is a strict complement but for thermal generators, it is both. This may matter for decisions such as entry, or for questions of competition policy and market manipulation.

Third, a storage unit starts empty and must always first buy energy. This is apparent in its profit function  $U_s$ . The total payoff in this simple example, is the discounted present value of the spread on energy sales  $(A - B)$  net of the first charge  $-B$ .

Finally, this example is the closest we can get to a benchmark: indeed, intertemporal arbitrage only makes sense in a dynamic model; there is no static equivalent.

## 4 Trading energy over the long horizon

As we know from the literature on repeated games and on stochastic games (see, for example, [Chatterjee et al. \(2003\)](#)), the game described in Section 2 admits a large number of equilibria. Short of constructing equilibria that exhibit features the analyst seeks, it is impossible to characterise an optimal strategy. But we wish to make progress to answer a practical question. To overcome this problem, we reduce the space of admissible strategies in two ways. First we focus on simple heuristics, which allows us to *compute* the value function of the storage operator for that heuristic. Then we can find the optimal level of this simple heuristic, and engage in comparative statics. In the example of Section 3, the heuristic is trivial: charge and discharge in full at any opportunity. In what follows, we explore richer heuristics, where capacity need not be used in full at every opportunity.

Second we must describe the equilibrium behavior of the other  $n$  players. We elect to restrict attention to the repetition of the Cournot equilibrium stage game – see Lemmata 1 and 2. This equilibrium delivers the lowest payoffs to sellers, and entails the least quantity distortions and the narrowest price spread. This equilibrium is simple to describe, unlike any of the more sophisticated equilibrium strategies one can construct. Our last justification

is the work of [Bonatti et al. \(2017\)](#), who study a dynamic Cournot model under incomplete information with learning. The equilibrium converges to the repeated static Nash equilibrium.

## 4.1 Independent binary shocks

We stay with the simple independent shock structure  $(1/2, 1/2)$  to begin with, which affords us some tractability, even though how much to (dis)charge is now endogenous. The objective of the storage operator is

$$\mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t p_t (s_t - b_t) \right], \quad (7)$$

subject to the law of motion (1) and the important capacity constraint

$$0 \leq c \leq k, \quad (8)$$

with corresponding value function

$$V(c) = \sup_{b,s} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t p_t (s_t - b_t) \right].^{14} \quad (9)$$

Then the recursive equation may be written in the following form:

$$\begin{cases} V(c) &= \frac{1}{2} \left( -\frac{1-a+b(c)}{n+1} \cdot b(c) + \beta V(c+b(c)) \right) \\ &+ \frac{1}{2} \left( \frac{1+a-\delta s(c)}{n+1} \cdot \delta s(c) + \beta V(c-s(c)) \right), \\ V(0) &= \frac{1}{2-\beta} \left( -\frac{1-a+b(0)}{n+1} \cdot b(0) + \beta V(b(0)) \right), \\ V(k) &= \frac{1}{2-\beta} \left( \frac{1+a-\delta s(k)}{n+1} \cdot s(k) + \beta V(k-s(k)) \right), \end{cases} \quad (10)$$

where (1) and (8) imply  $0 \leq b(c) \leq k-c$  and  $0 \leq s(c) \leq c$  for any  $c$ . The function  $b(c)$  is how much storage would like to buy if its state of charge is already  $c$ ; it is different from 0 only if the shock is negative. Likewise,  $s(c)$  is how much storage with its state of charge  $c$  would like to sell, which is relevant only under the positive shock. We aim to find functions

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<sup>14</sup>We dispense proving that the Dynamic Programming Principle holds in this environment, which is quite standard.

$b(c)$  and  $s(c)$  that maximize  $V(0)$  from time 0 and subject to (1) and (8). Given the nature of the stochastic process, it is immediate that  $V$  is time-invariant.

#### 4.1.1 Heuristic 1: proportional bids (constant fraction)

Our first heuristic calls for constant charging and discharging fractions. For example, starting from empty, if the chosen fraction is  $1/2$ , the storage operator charges  $k/2 = c$ , which is now the new state of charge. If facing another negative shock, she charges  $(k - c)/2 = k/4$ , now the state of charge is  $c = 3k/4$ . If facing a positive shock instead, she discharges  $c/2 = k/4$ , so the state of charge is  $c = k/4$ . And so on. More formally,  $b(c) = r(k - c)$  or  $s(c) = rc$ , respectively ( $0 \leq r \leq 1$ ). This problem is rendered complicated for two reasons. First, the constraint  $0 \leq c \leq k$  implies that, while the action (buy or sell) is governed by the stochastic shock, its quantum depends on the state  $c$ . Second, the grid of the state space grows exponentially (if  $r < 1$ ). In addition, a storage unit cannot start from an arbitrary state, but it must commence at  $c = 0$ . Hence it can be stuck at that level for some period before being able to charge and start trading. Nonetheless there exists a recursive structure that can be exploited. Let  $b(c) = r(k - c)$  and  $s(c) = rc$  and

$$B(rk) = \frac{1 - a + rk}{n + 1} \cdot rk, \quad A(rk) = \frac{1 + a - \delta rk}{n + 1} \cdot \delta rk.$$

**Proposition 3.** *The overall expected profit of storage is*

$$\begin{aligned} U_s^P &= \frac{1}{2[1 - (1 - r)\beta]} \left( -B(rk) + \frac{\beta r}{2(1 - \beta)} (-B(rk) + A(rk)) \right) \\ &+ \frac{\beta}{1 - \beta} \frac{k^2 r^3 (1 - r)(1 + \delta^2)}{4(n + 1)(1 - (1 - r)^2 \beta)}. \end{aligned} \quad (11)$$

The proof of this Proposition, as all others, is relegated to the Appendix, Section A.2.

Expression (11) entails three elements, abstracting from the multiplier that is a modified discount factor. The first term in the bracket is the cost of the initial charge. The second term is the discounted arbitrage profit from the first trade onward; this is the simple “buy low, sell high” mantra. The last term is strictly positive and not at all connected to arbitrage since it

is independent of  $a$ , and  $r$  is only linearly connected to  $k$  (so its optimum is independent of  $k$ ). It represents the benefit of flexibility in the face of uncertainty. Indeed, when  $r = 1$ , as in the introductory example of Section 3, this last term is zero, and all the weight is assigned to the arbitrage revenue. This term is largest for some interior value, which captures this notion of flexibility that arises from the asymptotic behaviour of this heuristic. That is, facing a series of identical shocks, for example, negative shock, the storage unit buys less and less energy. But upon a reversal, it sells a lot. We come back to this point later.

The payoff given in (11) is expressed in terms of the capacity  $k$  of the storage unit and its choice of heuristic  $r$ , as are the quantities  $A(rk)$  and  $B(rk)$ . This allows us not only to find the optimal proportion  $r$  to maximise  $U_s^P$ , but also to engage in comparative statics with respects to  $k$ . The first-order condition of (11) does not lend itself to easy manipulation nor interpretation, but Condition (11) can be graphed. To this end, let  $n = 2$ ,  $\delta = 0.95$ , and  $\beta = 0.95$ . Consider payoffs values for different capacities and shock magnitudes. Figure 1 corresponds to high shocks  $a = 0.6$  with capacity  $k$  moving from 0.15 to 1.15, and Figure 2 corresponds to low shocks  $a = 0.2$  where  $k$  changes from 0.05 to 0.3.

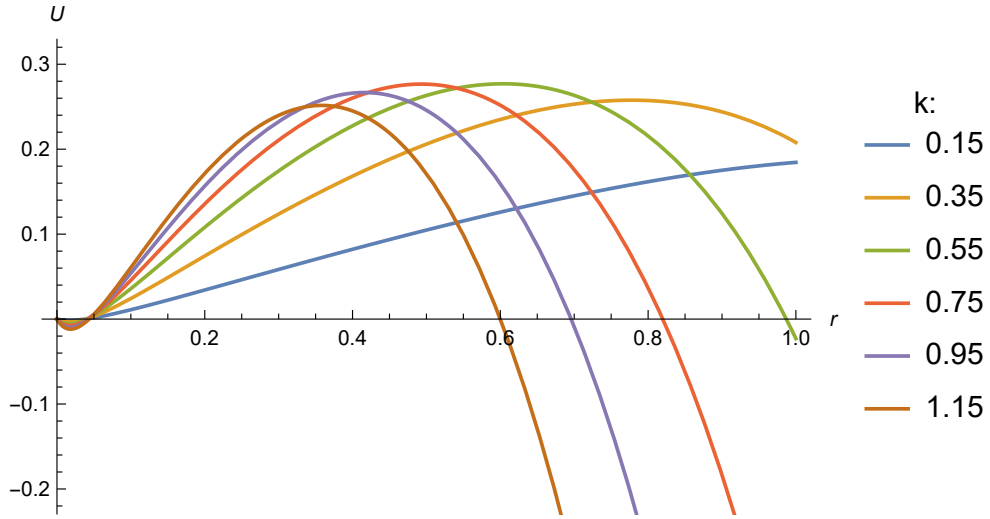


Figure 1: Payoff functions  $U_s^P(r)$  for different capacities  $k$  when the magnitude of the shock is high enough:  $a = 0.6$ .

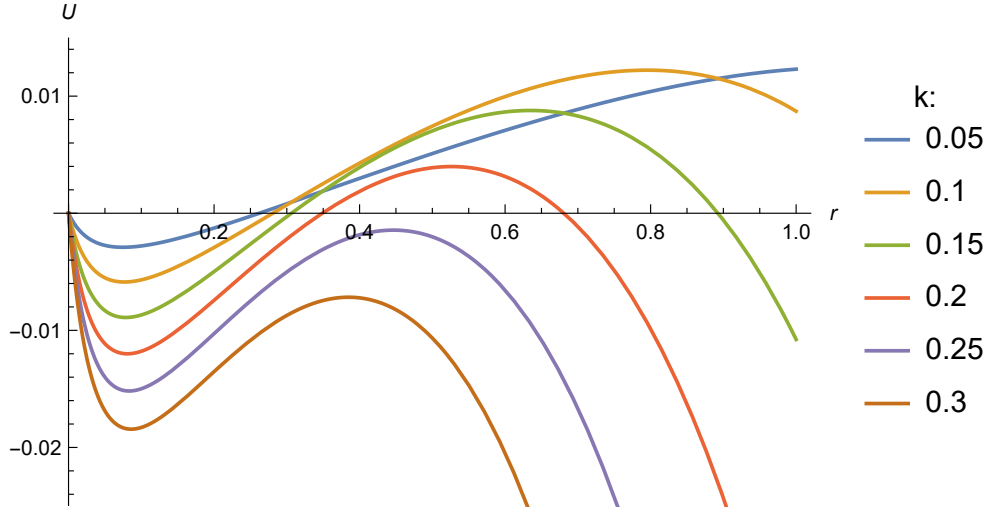


Figure 2: Payoff functions  $U_s^P(r)$  for different capacities  $k$  when the magnitude of the shock is low:  $a = 0.2$ .

First we see that for small relative capacity  $k/a$ , it is optimal to charge and discharge in full at each opportunity. This is the heuristic of our introductory example. It is easy to understand: with a small capacity, the storage unit cannot wield much market power, so the arbitrage spread is not eroded by the full use of capacity. Second, from the lowest relative capacity, the maximum of the payoff function  $U_s^P$  increases as capacity expands, however only to a point. Third, concurrently, as relative capacity increases, the optimal proportion  $r$  decreases: large storage units use less of their capacity in any single trade. This is apparent from (11), where the arbitrage term  $(A - B)$  rapidly decreases in  $k$ .

We also see that very small values of  $r$  deliver no surplus at all even though there are no fixed costs; this is apparent from (11) as well: at  $r$  close to zero, there is neither arbitrage nor flexibility. These “frictions” can be explained too: when  $r$  is very small, the quantities traded keep decreasing rapidly and become negligible in finite time. This nullifies the arbitrage spread  $A(rk) - B(rk)$ , and the continuation value *after* the first charge rapidly becomes negligible, but that first charge is a cost.

Finally, very small shocks cannot sustain the operations of a storage unit, as we see from Figure 2. For a relatively large capacity, a storage unit can only make losses – and so should not enter the market. This is easy to understand: with small shocks  $\varepsilon$ , the spread can only be small. Any quantity bought or sold by the storage unit erodes it, and a (relatively) large

capacity easily nullifies that spread for any choice of  $r$ .<sup>15</sup> This can also be seen from the profit function (11), which is linear in  $a$  but quadratic (negative) in  $k$ .

There are considerable subtleties to these findings, the discussion of which we postpone until after presenting our second heuristic.

#### 4.1.2 Heuristic 2: constant quantities

Here the storage operator buys or sells a constant quantity  $X$  (e.g 10MW) each period, starting from empty as well. To avoid having to deal with partial fills at the boundaries, we let  $X := k/m$ , so  $m$  is the number of steps to move from empty to full. In the set of admissible strategies, restricting  $m$  to be an integer may not be fully optimal, but we expect the corresponding loss to be small – if it exists.<sup>16</sup>

The constraints at 0 and at  $k$  matter even more here than in the proportional case. Under the proportional heuristic, boundaries are never reached: the storage unit can never be completely full nor completely empty in finite time. But here it becomes completely empty or completely full with positive probability. This induces rich dynamics that we label “waves”. Handling these waves is the main challenge in this otherwise simple environment. First let

$$B\left(\frac{k}{m}\right) = \frac{1 - a + k/m}{n + 1} \cdot \frac{k}{m}, \quad A\left(\frac{k}{m}\right) = \frac{1 + a - \delta k/m}{n + 1} \cdot \delta \frac{k}{m}.$$

Now consider a standard binomial tree representing the state space as drawn in Figure 3.

---

<sup>15</sup>Even though the relative capacity  $k/a$  is almost constant for each choice  $k$  across Figures 1 and 2.

<sup>16</sup>This is actually duly verified in Section 4.1.4.

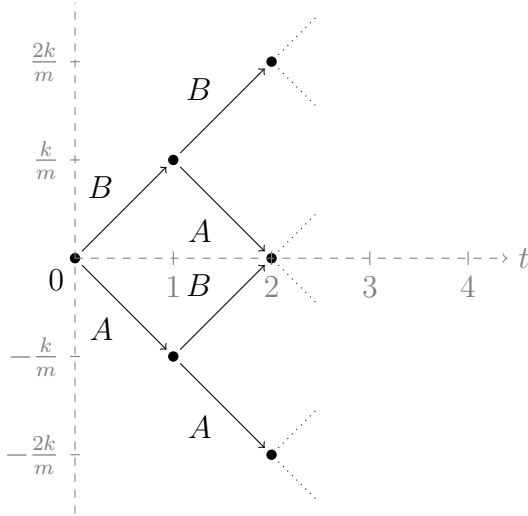


Figure 3: No boundaries.  $A$  and  $B$  available in any point.

Starting from zero, truncate this binomial tree from below – this leaves the top half of the tree, as in Figure 4. In this Figure, the light gray area is the region where the probability weights that cannot go down start going up instead. This changes the probabilities of reaching any node; for example, the point with coordinates  $(1,0)$  can be reached from the preceding node  $(0,0)$ , whereas in unconstrained the binomial tree (Figure 3), it can *never* be reached. Likewise for the point  $(2,1)$ , which can be reached from  $(1,0)$  in the truncated tree but never in the unconstrained tree. In turn this affects the probability of reaching  $(3,1)$ , which is accessible in both cases. The states with affected probabilities are marked with a thicker dot.

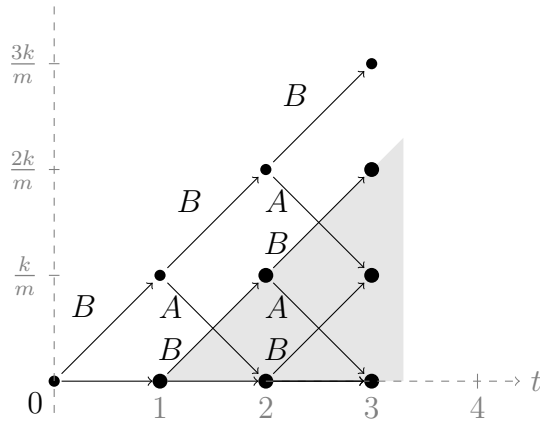


Figure 4: One bound – 0. There is no  $A$  in state  $c = 0$ .

Then truncate this tree further from above at the capacity level  $k$  to create a tunnel. This is depicted in Figure 5. The admissible state space is limited to that tunnel, in which we already know either new states can be reached, or some states can be reached with different



probabilities. This upper boundary modifies the state space in much the same way, but in the other direction. Hitherto unreachable states can be reached, and the probability mass on already reachable states can be changed. In Figure 5 the darker gray area represents a second region, in which the upper bound  $k$  becomes active and forces the probability mass back down; hence the waves. This process continues on over the infinite horizon, and the successive reflections at the boundaries perpetuate these waves.

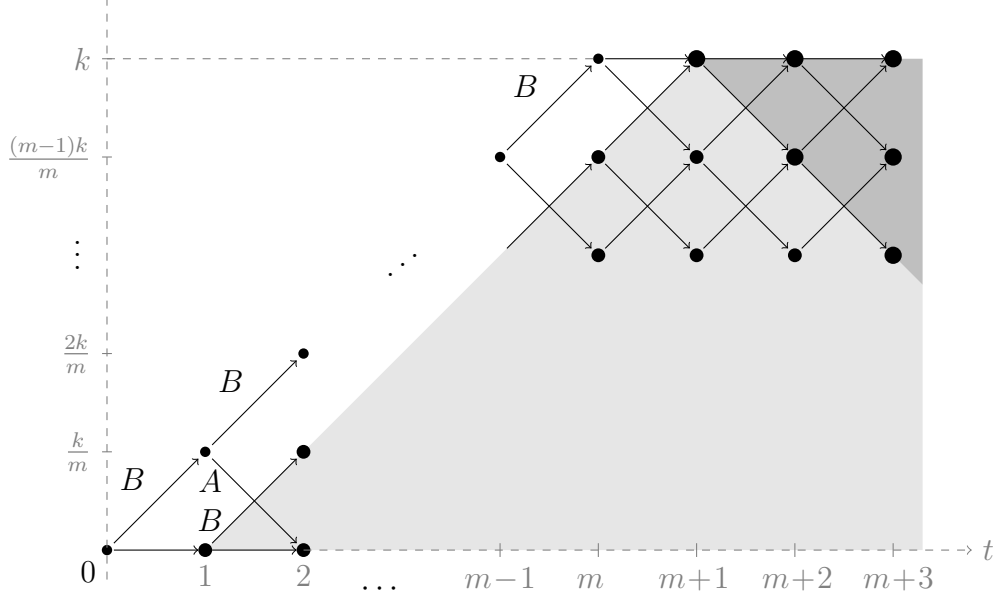


Figure 5: Two bounds – 0 and  $k$ . Here there are three types of thickness of points (states), depending on how many waves affect the corresponding probability (here, 0, 1, or 2). There is no  $A$  in state  $c = 0$  and no  $B$  in state  $c = k$ .

These waves are periodic, which suggests a recursive structure can be uncovered. We are able to exploit this and compute the value function of the storage operator for this heuristic too.

**Proposition 4.** *The overall expected profit of the storage operator is*

$$\begin{aligned}
 U_s^C &= \frac{-B(k/m) + \beta A(k/m)}{2(1-\beta)} - \frac{A(k/m)}{\beta} \sum_{i=1}^{\infty} \left(\frac{\beta}{2}\right)^{2i} \left(C_{2i-1}^{i-1} + \frac{\beta}{2} C_{2i}^i\right) \\
 &+ \frac{1+\beta}{\beta} \left( B\left(\frac{k}{m}\right) \sum_{j=0}^{\infty} \left(\frac{\beta}{2}\right)^{(m+1)(2j+1)} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} C_{2i+(m+1)(2j+1)}^i \right. \\
 &\quad \left. - A\left(\frac{k}{m}\right) \sum_{j=1}^{\infty} \left(\frac{\beta}{2}\right)^{2(m+1)j} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} C_{2i+2(m+1)j}^i \right). \tag{12}
 \end{aligned}$$

Using the formula

$$\sum_{i=0}^{\infty} \beta^i C_{2i+r}^i = \frac{2^r}{\sqrt{1-4\beta} \left(1 + \sqrt{1-4\beta}\right)^r}$$

from [Graham et al. \(1994\)](#) (p. 203) and introducing the new discounting coefficient

$$\tilde{\beta} = \frac{\beta}{1 + \sqrt{1 - \beta^2}},$$

(12) rewrites more compactly as

$$\begin{aligned} U_s^C &= \frac{1}{2(1-\beta)} \left( -B \left( \frac{k}{m} \right) + \tilde{\beta} A \left( \frac{k}{m} \right) \right. \\ &\quad \left. - \frac{2\sqrt{1-\beta^2}}{\beta} \frac{\tilde{\beta}^{m+1}}{1 - \tilde{\beta}^{2(m+1)}} \left( -B \left( \frac{k}{m} \right) + \tilde{\beta}^{m+1} A \left( \frac{k}{m} \right) \right) \right). \end{aligned} \quad (13)$$

In (12), the  $C_n^l$  terms are binomial coefficients. Of course, (13) is a lot easier to understand. This payoff function contains two parts – ignoring the multiplier, which is a simple discount factor. The first part  $-B + \tilde{\beta}A$  captures the payoff that accrues from the stochastic process depicted in Figure 4. It is the sum of the risk-free arbitrage spread  $-B + A$  (again, the mantra “buy low, sell high”), which results from the unconstrained process represented in Figure 3, and having to forego the “short-selling” revenue  $-(1 - \tilde{\beta})A$  because of the constraint  $0 \leq c$ . The second part is the cost of observing the initial constraint  $c \leq k$ , and all the subsequent repetitions of the constraints  $0 \leq c \leq k$  that follow because of the waves. This last term embeds what we call the “continuation risk”: the storage operator may keep buying energy and fail to sell for a long time, which renders the operation unprofitable. Here the constraint  $c \leq k$  actually helps: it caps losses from buying energy “forever” – more precisely, for a long time before selling it. This last term also shows a trade-off in the number of steps  $m$  to charge and discharge: for large enough  $m$ ,  $-B + \tilde{\beta}^{m+1}A$  turns negative, but then the multiplier in front of that bracket converges to zero. Too few steps are bad, and so are too many; indeed in Figure 6,  $m$  is clearly interior.

The payoff function (13) thus differs from (11) for with proportional bids, the boundaries are *never* reached (once the lower boundary has been exited). Furthermore, the con-

tinuation risk is milder with proportional bids thanks to the asymptotic behaviour  $r < 1$  induces.

As with the proportional case, we graph this function using the same parameters as before. The dots on the curves correspond to the actual choices that are possible – a fixed quantity, e.g. 0.5. Let  $n = 2$ ,  $\delta = 0.95$ , and  $\beta = 0.95$ . Fig. 6 shows high shocks  $a = 0.6$  with capacity  $k$  moving from 0.15 to 1.15, and Fig. 7 depicts low shocks  $a = 0.2$  where  $k$  changes from 0.05 to 0.35.

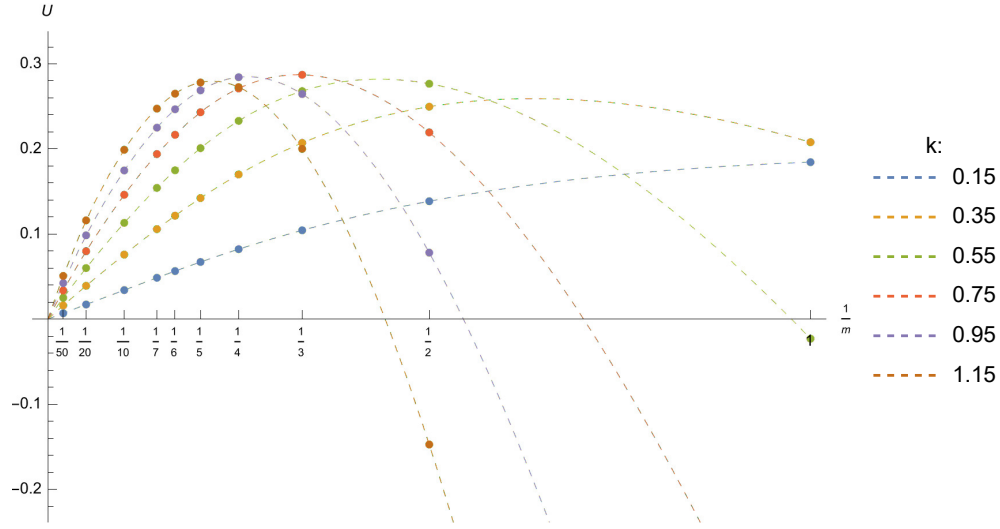


Figure 6: Payoff functions  $U_s^C(r)$  for different capacities  $k$  when the magnitude of the shock is high enough:  $a = 0.6$ .

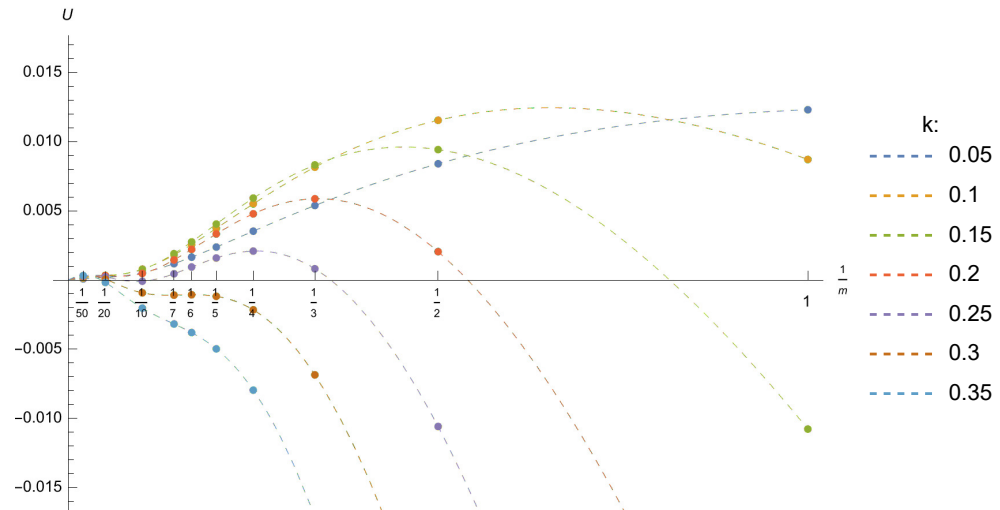


Figure 7: Payoff functions  $U_s^C(r)$  for different capacities  $k$  when the magnitude of the shock is low:  $a = 0.2$ .

Overall Figures 6 and 7 complement nicely Figures 1 and 2; they simultaneously enrich them, and confirm the overall message. That is, the optimal quantity choice is interior except for a

very small capacity, and rapidly much less than  $k$  as capacity increases. In turn the optimal capacity choice is also interior so as to not wipe out the arbitrage spread with the continuation risk. Finally, too small a shock induces too small a spread, which cannot sustain storage.

### 4.1.3 The main point

Up to some details, the two heuristics we study convey the same message. Fix a capacity, the storage operator is a monopolist on its arbitrage opportunities and behaves as such. Its profit function is concave in the variable of interest (be it  $r$  or  $m$ ) and it internalises the well-known trade-off between the extensive margin and the intensive margin. This is the usual market power effect.

Things are quite different when it comes to capacity – even if not quite a choice variable but rather a parameter. In brief, a larger capacity allows the storage unit to better manage the “continuation risk”: the risk of being stuck at (or arbitrarily close to) a boundary. But too large a capacity *exacerbates* the *cost* of being stuck: indeed, the storage unit can find itself repeatedly buying and then unable to sell for a long time. There is a trade off there too. In other words, more capacity allows for more flexibility in the choice of quantities traded to delay hitting the boundary – for example,  $m = 2$ ,  $m = 3$ ,  $m = 4$  in Figure 6. But beyond a level, it only increases the cost at the boundary ( $m = 6$ ,  $m = 7$ ), for then storage takes too long to revert its position. We emphasize the continuation risk is moot absent market power, for then the arbitrage gain dominates and the unit charges and discharges in full at every opportunity.

To validate this interpretation, in Section A.3 of the Appendix we compute the quantities corresponding to each  $m$  (and  $r$ ); they increase slightly for the proportional heuristic, but are almost constant for the second one. Continue with Figure 6: the choice of  $m = 2$  for  $k = 0.35$  is improved upon by a quantity increase for  $m = 3$  when  $k = 0.55$ , but the quantity corresponding to  $m = 5$  when  $k = 0.95$  is essentially the same. So why is the payoff lower then? It cannot be the impact of quantities; rather it is the continuation risk, the cost of which now dominates.

The skeptic may argue the heuristics we study are too rigid in that they commit the storage unit to the same action regardless of the state of charge  $c$ . Why keep charging as one approaches full charge? First, this is strictly true only under the constant-bid heuristic; under proportional bids,  $r$  is constant but the quantities traded keep declining. Second, under a less committal strategy, a large-capacity unit could stop charging after some steps and wait for a reversal – that is, stop increasing its cost. But this is exactly the same as choosing a smaller capacity. That is, the optimal capacity remains interior.

Finally, previewing Section 4.3 in which we allow for Markovian shocks of arbitrary persistence, the continuation risk vanishes as persistence decreases: there is (almost) no risk of being stuck at a boundary.

#### 4.1.4 Robustness check: a more general version of constant bids

One can wonder how much is lost from not only selecting a heuristic such as constant bids, but also from simplifying that heuristic to factors of  $k$  – i.e.  $X = k/m$ . In this section we relax this simplification (so,  $X \neq k/m$ ). In Figure 8 we show the event tree for the case  $k/2 < X < k$ .

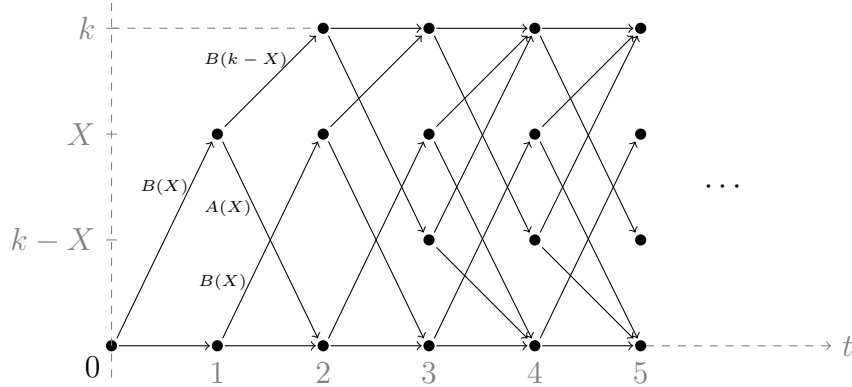


Figure 8: All trajectories in the first five periods across four possible states of charge when  $k/2 < X < k$ .

We see that even with this more flexible structure (and  $k/2 < X < k$ ), there are only four possible states of charge: 0 (empty),  $X$ ,  $k - X$ , and  $k$  (full). The main difference from the previous case is that when storage is in state of charge  $X$  and faces a negative shock again, it cannot buy  $X$  units of energy again. Instead, it has to buy the remainder to its full capacity,

which is  $k - X < X$ . So, starting from empty (0), the first step is “large” and if charging a second time it is smaller. The same thing works when the storage unit only holds charge  $k - X$  and faces one more consecutive positive shock. Then it can sell only the remaining capacity  $k - X$  of energy to become completely empty.

A finer structure allowing for more steps has  $k/(m+1) < X < k/m$ ,  $m \geq 1$ , and we immediately see from Figure 8 that for any  $m \geq 1$ , only the last step in either direction may be curtailed – just as for two steps. We are able to obtain expressions for the storage payoffs for  $k/3 < X < k/2$ ,  $k/4 < X < k/3$ , and  $k/5 < X < k/4$ . Analytical results for smaller  $X$  (that is,  $m \geq 5$ ) are out of reach because of the expanding size of the transition matrix.

**Proposition 5.** *Assume the storage unit sets constant bids  $X$  ( $k/(m+1) < X < k/m$ ) when it buys under any state of charge different from  $mX$  or when it sells under any state of charge different from  $k - mX$ . Otherwise, the storage unit bids  $k - mX$  when it either buys under state  $mX$  or sells under state  $k - mX$ . The expected profit of storage reads:*

1. For  $1/2 < X < 1$  ( $m = 1$ ):

$$U_s^X = -\frac{1}{2} \left( B(X) + \frac{\beta}{2-\beta} B(k-X) \right) + \frac{\beta}{4} \left( D(X) + \frac{\beta^2}{4-\beta^2} D(k-X) \right); \quad (14)$$

2. For  $1/3 < X < 1/2$  ( $m = 2$ ):

$$\begin{aligned} U_s^X = & -\frac{1}{2-\beta} \left( B(X) + \frac{\beta^2}{2(2-\beta^2)} B(k-2X) \right) \\ & + \frac{\beta}{4-\beta^2} \left( D(X) + \frac{\beta^4}{8(2-\beta^2)} D(k-2X) \right); \end{aligned} \quad (15)$$

3. For  $1/4 < X < 1/3$  ( $m = 3$ ):

$$\begin{aligned} U_s^X = & -\frac{1}{2(2-\beta^2)} \left( (2+\beta)B(X) + \frac{\beta^3}{4-2\beta-\beta^2} B(k-3X) \right) \\ & + \frac{\beta}{8(2-\beta^2)} \left( (4-\beta^2)D(X) + \frac{\beta^6}{(4-\beta^2)^2-4\beta^2} D(k-3X) \right); \end{aligned} \quad (16)$$

4. For  $1/5 < X < 1/4$  ( $m = 4$ ):

$$\begin{aligned}
U_s^X = & -\frac{1}{4 - 2\beta - \beta^2} \left( 2B(X) + \frac{\beta^4}{2(4 - 3\beta^2)} B(k - 4X) \right) \\
& + \frac{\beta}{(4 - \beta^2)^2 - 4\beta^2} \left( 2(2 - \beta^2)D(X) + \frac{\beta^8}{4(4 - \beta^2)(4 - 3\beta^2)} D(k - 4X) \right);
\end{aligned} \tag{17}$$

where  $D(X)$  is the NPV of the arbitrage spread:

$$D(X) = \frac{A(X) - B(X)}{1 - \beta}.$$

Echoing the introductory example, the first term in all these formulae is the initial purchase cost, and the second one is the arbitrage spread. Both are suitably discounted.

#### 4.1.5 Comparing heuristics

In Figures 9 and 10 we directly compare the performance of the heuristics we study and of the more flexible approach (still restricted to  $1/5 < X < 1$ ) of the preceding section. Red represents linear bids, blue stands for constant bids with integer  $m$ , and purple shows the payoffs from bids for any  $X \in (k/5, k)$ . The graphs of these payoff functions feature kinks at each of the integer  $m$  because altering  $m$  amounts to altering the regime under which the storage unit operates. The first series is concerned with relatively large shocks ( $a = 0.6$ ), and the second one with small shocks ( $a = 0.2$ ).

First, it is difficult to rank the two main heuristics. While constant quantities do not systematically dominate proportional bidding, the maximum always exceeds – at least weakly – the maximum achieved under the proportional rule, but this may be sensitive to the parameter values we use. We can also see that as capacity  $k$  increases, the linear-bid heuristic performs better. This improvement stems from the more flexible nature of this heuristic, which never reaches the boundaries for any  $r < 1$  and commits the storage operator to either buy ever smaller quantities (in a sequence of negative shocks), or conversely. Further, for  $r$  large enough, as soon as reversal does occur, the quantity sold at the first positive shock (or bought at the first negative shock), is large. This is an approach that is both more prudent and more

flexible, which is comparatively better when capacity is large. Indeed, for large shocks and large capacity (the four bottom panels of Figure 9), the maximizer under the proportional rule lies to the right of the maximizer under constant bids, and the constant-bid heuristic clearly dominates for (many) small steps. A large  $m$  delays reaching a boundary at which the storage unit may be stuck. Second, under constant quantities, a storage unit is immune to the (small) losses that accrue under the proportional heuristic when the quantities are very small. Under constant bids, the quantities never become vanishingly small, so the arbitrage revenue is never negligible. Finally we can see that large shocks are completely essential to profitable trading, as seen in Figure 10. There a large capacity is of little use.

Of course these differences in behaviour stem from the difference in the stochastic process that is induced by the choice of heuristic. That is, the heuristics interact with the exogenous stochastic process to define an *endogenous* process; this is what defines a stochastic game Shapley (1953). Under the proportional heuristic, the capacity constraint induces an asymptotic path but the binomial tree itself is never truncated. This is what makes it more flexible than the constant-quantity approach, which modifies the binomial tree outright and generates reflections.

From Figure 9 we see that, except for very small capacity  $k$  (in which case  $m = 1$  is optimal), relaxing the integer constraint and allowing  $X \in (k/5, k)$  (in purple) delivers a modest improvement compared to  $X \in \{k/m, m \in \mathbb{N}\}$ . For the majority of cases it finds a new optimum that beats the other two heuristics – for example,  $2 < m < 3$  for  $k = 0.65$ .



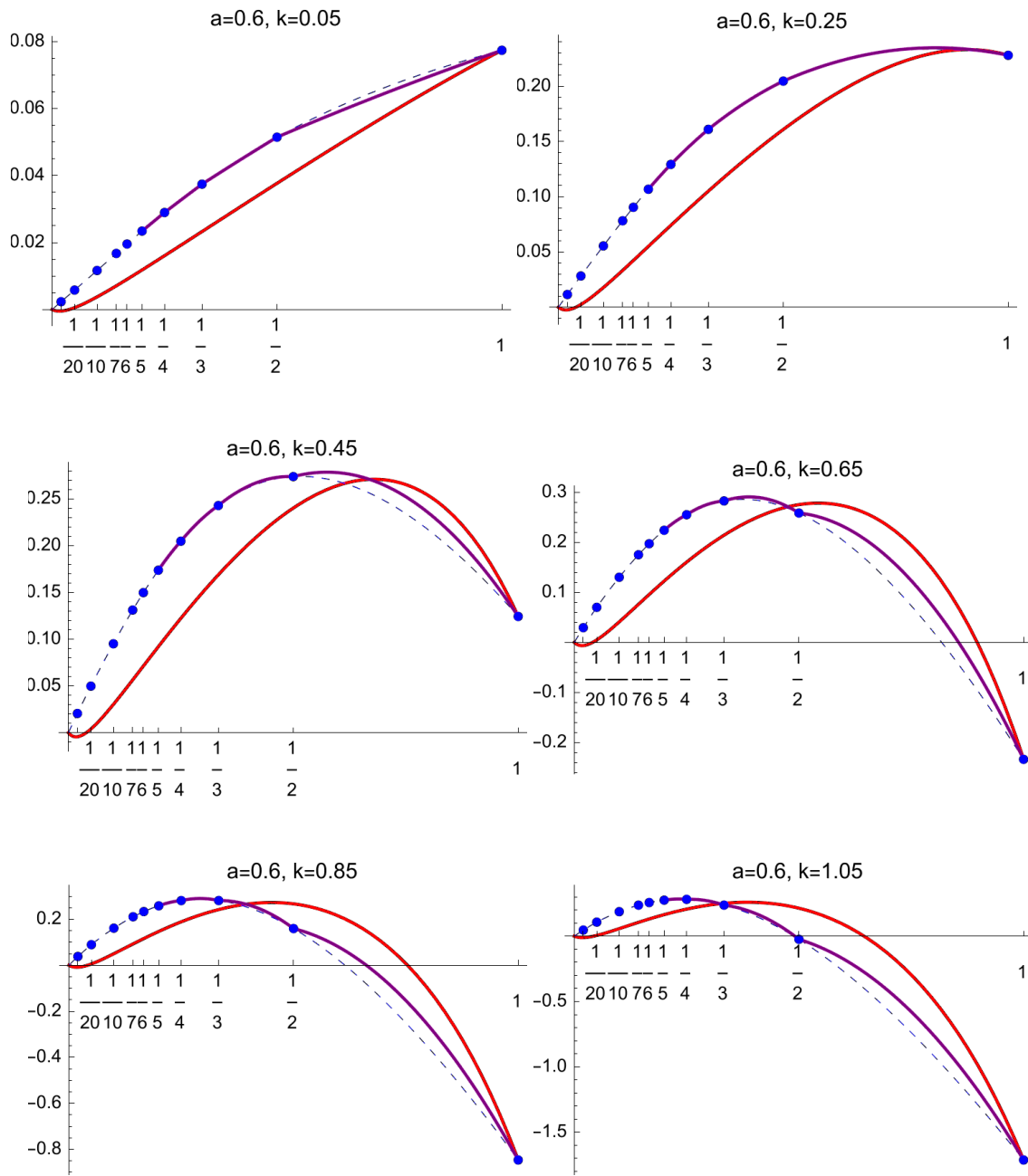


Figure 9: Both linear and constant bids payoffs in the same graphs for different  $k$  and  $a = 0.6$ .

Finally, we remark that this more flexible heuristics approximately behaves like some kind of combination of the constant bid and proportional bid approaches. This robustness check gives us comfort in thinking that the heuristics we study operate “not far” from the strictly optimal strategy.



**Proposition 6.** *Assume that*

$$\frac{(1+a)^2(n+1)^2}{4n(1+a^2) + (1+a)^2(n+1)^2} \leq \beta \leq \frac{2}{1 + \frac{2\delta(1+a-\delta k)}{(n+1)(1-a+2k)}}. \quad (18)$$

*Then there exists a dynamic Subgame Perfect Nash Equilibrium, such that:*

- *in each period, the generators set quantities*

–  $\varepsilon = -a$ :

- $q^* = \frac{1-a}{2n}$  *if none of the generators deviated in the previous rounds,*
- $q^* = \frac{1-a+k}{n+1}$  *if storage is empty and any of the generators deviated in the previous rounds,*
- $q^* = \frac{1-a}{n+1}$  *if storage is full and any of the generators deviated in the previous rounds;*

–  $\varepsilon = a$ :

- $q^* = \frac{1+a}{2n}$  *if storage is empty and none of the generators deviated in the previous rounds,*
- $q^* = \frac{1+a}{n+1}$  *if storage is empty and any of the generators deviated in the previous rounds,*
- $q^* = \frac{1+a-\delta k}{n+1}$  *if storage is full;*
- *storage does not enter the market if none of the generators have deviated; otherwise, storage enters the market if*

$$B < \frac{\beta}{2-\beta}A. \quad (19)$$

*Cumulative consumers' expected payments per period  $C^0$  are*

$$C^0 = \frac{1+a^2}{4}.$$

*Expected payoffs  $U_g^0$  and  $U_s^0$  of the generators and the storage unit, respectively, take the*

following form:

$$U_g^0 = \frac{1 + a^2}{4(1 - \beta)n}, \quad U_s^0 = 0.$$

In this equilibrium, generators collude to the joint-profit maximising quantities and the storage unit never charges. Equilibrium play must deter two kinds of deviations. First, the generators must elect to not deviate; this is supported by the threat of Cournot reversion, which is subgame perfect. Second, the storage operator also prefers not charging, otherwise the punishment reverts to the equilibrium described in the introductory example, which is therefore also subgame perfect. This threat is sufficient because the charging cost is too high compared to what the storage unit can collect once players enter the punishment phase. That is, the strategic effect working through market power supports the equilibrium. Absent market power this equilibrium does not exist.

Storage is excluded because it starts empty and must first charge to become active. This is an important consideration that is not studied to its full extent in the works of [Andres-Cerezo and Fabra \(2023\)](#), where exclusion is not considered. It is also irrelevant in the model of [Butters et al. \(Working Paper\)](#), where storage is assumed to behave competitively, in which case the spread is not sensitive to market power. This result makes it plain that starting from empty is not just costly for the storage operator; it can be socially costly as well since storage if welfare enhancing in this model. Exclusion may thus be overcome with the help of a small subsidy. Here it is enough to cover the first charge to not only foster storage activity, but also to unravel the collusive equilibrium.

### 4.3 A richer Markovian structure

The payoff functions we can compute in Section 4.1 feature a cost of uncertainty – see equations (11) and (12). This cost stems from the risk of facing multiple negative shocks, which induce an incentive to charge, but being already fully charged, and conversely.

A glance at the payoff functions (11) and (12) suggests that a sequence of perfectly negatively correlated shocks  $-a, a, -a, a, \dots$  would deliver the highest (and certain) payoff. In this

Section we relax the strict independence assumption and investigate the constant-quantity heuristics when shocks follow a non-degenerate Markov chain, and can be made either not persistent or to carry significant persistence. The goal is to better understand the impact of risk on the behavior of the storage operator.

To this end, we consider a case where the storage buys and sells its capacity in any of one step ( $m = 1$ ), two steps ( $m = 2$ ) or three steps ( $m = 3$ ); richer heuristics, with  $m > 3$  are no less interesting but beyond what we can manage. With  $m = 2$ , for example, there are four possible states of charge that are payoff relevant: an empty unit, a half-full unit after the negative shock, a half-full unit after the positive shock, and finally a full storage unit. There is no need to distinguish the nature of the shock at the boundaries because states 0 and  $k$  are accessible only after positive and negative shocks, respectively. Assume now that shocks  $\varepsilon_t$  form a discrete-time Markov chain:

$$\begin{aligned}
Pr\{\varepsilon_0 = a\} &= x, & Pr\{\varepsilon_0 = -a\} &= 1 - x, \\
Pr\{\varepsilon_{t+1} = a | \varepsilon_t = a\} &= x, & Pr\{\varepsilon_{t+1} = -a | \varepsilon_t = a\} &= 1 - x, \\
Pr\{\varepsilon_{t+1} = a | \varepsilon_t = -a\} &= 1 - y, & Pr\{\varepsilon_{t+1} = -a | \varepsilon_t = -a\} &= y
\end{aligned} \tag{20}$$

for any  $t \geq 0$ . We denote the transition matrix by  $Q$  and its determinant by  $d$ :

$$Q = \begin{pmatrix} x & 1 - x \\ 1 - y & y \end{pmatrix}, \quad d = \det Q = x + y - 1,$$

and we let the functions  $A(k/m)$  and  $B(k/m)$  be defined as before.

**Proposition 7.** *Under conditions laid out below, there exists a dynamic equilibrium, such that*

- *the storage unit buys  $k/m$  under the negative shock until it reaches capacity  $k$  and sells  $\delta k/m$  under the positive shock until it becomes empty;*
- *in each period, the generators set quantities  $q^*$  according to static Cournot competition*

and based on the current shock and the state of the storage (full or empty). Namely,

$$\begin{aligned}
q^* &= \frac{1-a}{n+1} \quad \text{if storage is full and } \varepsilon = -a, \\
q^* &= \frac{1-a+k/m}{n+1} \quad \text{if storage is not full and } \varepsilon = -a, \\
q^* &= \frac{1+a}{n+1} \quad \text{if storage is empty and } \varepsilon = a, \\
q^* &= \frac{1+a-\delta k/m}{n+1} \quad \text{if storage is not empty and } \varepsilon = a.
\end{aligned}$$

1.  $m = 1$ ; this equilibrium exists if

$$B < \frac{\beta(1-y)}{1-\beta y} A. \quad (21)$$

and the expected payoff of the storage operator  $U_s^1$  takes the following form:

$$U_s^1 = \frac{1-x}{(1-\beta)(1-\beta d)} (-B + \beta(1-y)A + \beta y B).$$

2.  $m = 2$ ; this equilibrium exists if

$$B \left( \frac{k}{2} \right) < \frac{\beta(1-y)(1+\beta^2 d)}{1-\beta^2(1-x+yd)} A \left( \frac{k}{2} \right), \quad (22)$$

and the expected payoff of the storage operator  $U_s^2$  takes the following form:

$$U_s^2 = \frac{1-x}{(1-\beta)(1-\beta d)} \left( -B \left( \frac{k}{2} \right) + \beta(1-y)A \left( \frac{k}{2} \right) + \beta^2 y \frac{yB \left( \frac{k}{2} \right) + \beta x(1-y)A \left( \frac{k}{2} \right)}{1-\beta^2(1-x)(1-y)} \right).$$

3.  $m = 3$ ; this equilibrium exists if

$$B \left( \frac{k}{3} \right) < \beta(1-y) \frac{(1-\beta^2(1-x)(1-y))^2 + \beta^2 xy (1+\beta^2(xy-2(1-x)(1-y)))}{(1-\beta^2(1-x)(1-y))^2 - \beta^3 y^3 - \beta^4 xy(1-x)(1-y)} A \left( \frac{k}{3} \right), \quad (23)$$

and the expected payoff of the storage operator  $U_s^3$  reads

$$U_s^3 = \frac{1-x}{(1-\beta)(1-\beta d)} \left[ -B\left(\frac{k}{3}\right) + \beta(1-y)A\left(\frac{k}{3}\right) + \beta^3 y \frac{y^2 B\left(\frac{k}{3}\right) + x(1-y)(1+\beta^2 d)A\left(\frac{k}{3}\right)}{(1-\beta^2(1-x)(1-y))^2 - \beta^4 xy(1-x)(1-y)} \right].$$

When there is little persistence, the probability of being stuck at either boundary 0 or  $k$  is small; that is, the storage operator is almost guaranteed to reverse direction – for example, sell after charging. Compared to Section 4.1.2, the “waves” unfold faster. In turn, this stokes the incentives to charge in the first place, and so on. These rapid cycles reduce the uncertainty, but not the *volatility*; in fact, certain volatility is best for the storage operator. Low persistence effectively negates the continuation risk.

More generally, the payoffs  $U_s^m$ ,  $m = 1, 2, 3$  include two terms: the first one is  $-B(k/m) + \beta(1-y)A(k/m)$ , which is, modulo a multiplier, the discounted payoff from charging and discharging every other period. This is a storage unit operating under perfect foresight. The second term captures the cost of uncertainty, as best-responded to by the storage operator. Here the parameters and the best reply interact richly, as we show next.

In Figures 11 and 12 we plot the payoff functions of the storage operator projected on the dimensions  $x$  and  $y$ , which denote persistence, and with  $x = y$ . The red stands for the payoff function when  $m = 1$ , the blue for  $m = 2$  and the green for  $m = 3$ , all for the constant-quantity heuristic. All other parameters remain as in the other plots. Low persistence is clearly better in this environment, but we note that in some cases a very high persistence seems to improve payoffs over a moderate persistence. With higher persistence there are fewer cycles, each of which induces some losses; high persistence delays the onset of each these cycles.

When capacity is relatively small (compared to the shock), it is best to charge and discharge in full ( $m = 1$ ) for almost any persistence. The reason is that the storage operator has no (significant) market power, so there is no (significant) price impact. But when capacity increases, we observe more mixed results. First, as persistence increases, flexibility becomes valuable: charging and discharging in two steps and three steps starts dominating. It is better

able to cope with the uncertainty of the shocks. Second, with a large(r) capacity, restraint also pays off: charging and discharging in two steps (blue) dominates one step (red) for any persistence; and three steps (green) dominates both. It is best to not use the capacity in full at any point in time because of the market power effect, and this is exactly what  $m = 3$  delivers. These conclusions are replicated, but even starker, when shocks and capacity are even smaller. Then, in some cases, the three-step strategy is the only one that can deliver any positive surplus.

With this simple structure it is difficult to speak of the impact of high persistence in greater detail. With a lot of persistence in shock ( $x \rightarrow 1$  or  $y \rightarrow 1$ ), the storage operator can spend a lot of time at either boundary (0 or  $k$ ); if that is the case, one can conjecture she would like to buy or sell over many periods (so,  $m$  be large). We cannot treat this case, and aside from a strict numerical treatment, there is no hope of doing so because even a computer cannot find the eigenvalues of the matrices of interest.



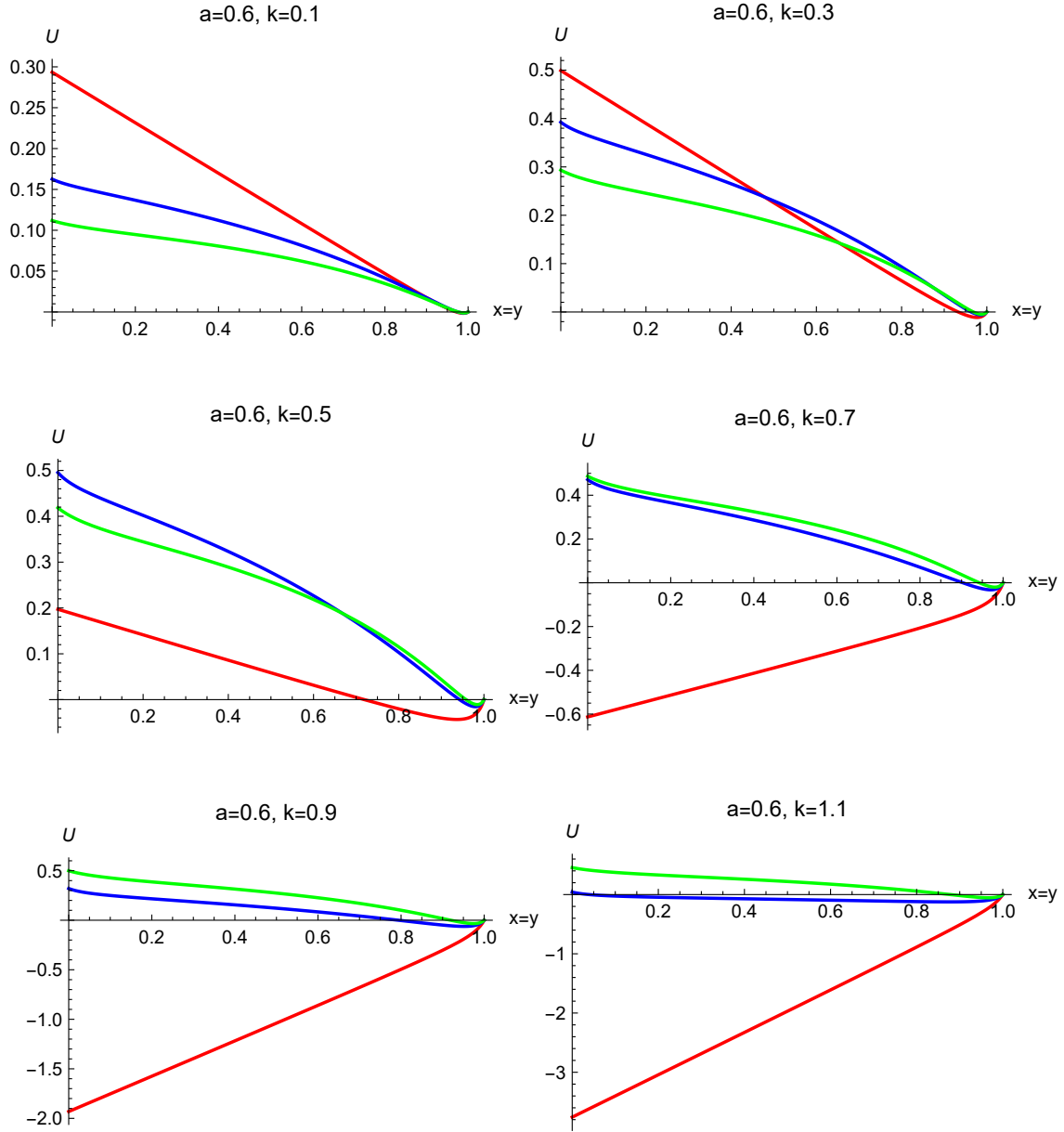


Figure 11: Payoffs under symmetric Markov shocks for divisible (green for  $k/3$  and blue for  $k/2$ ) and indivisible (red) capacities in the same graphs for different  $k$  and  $a = 0.6$ .

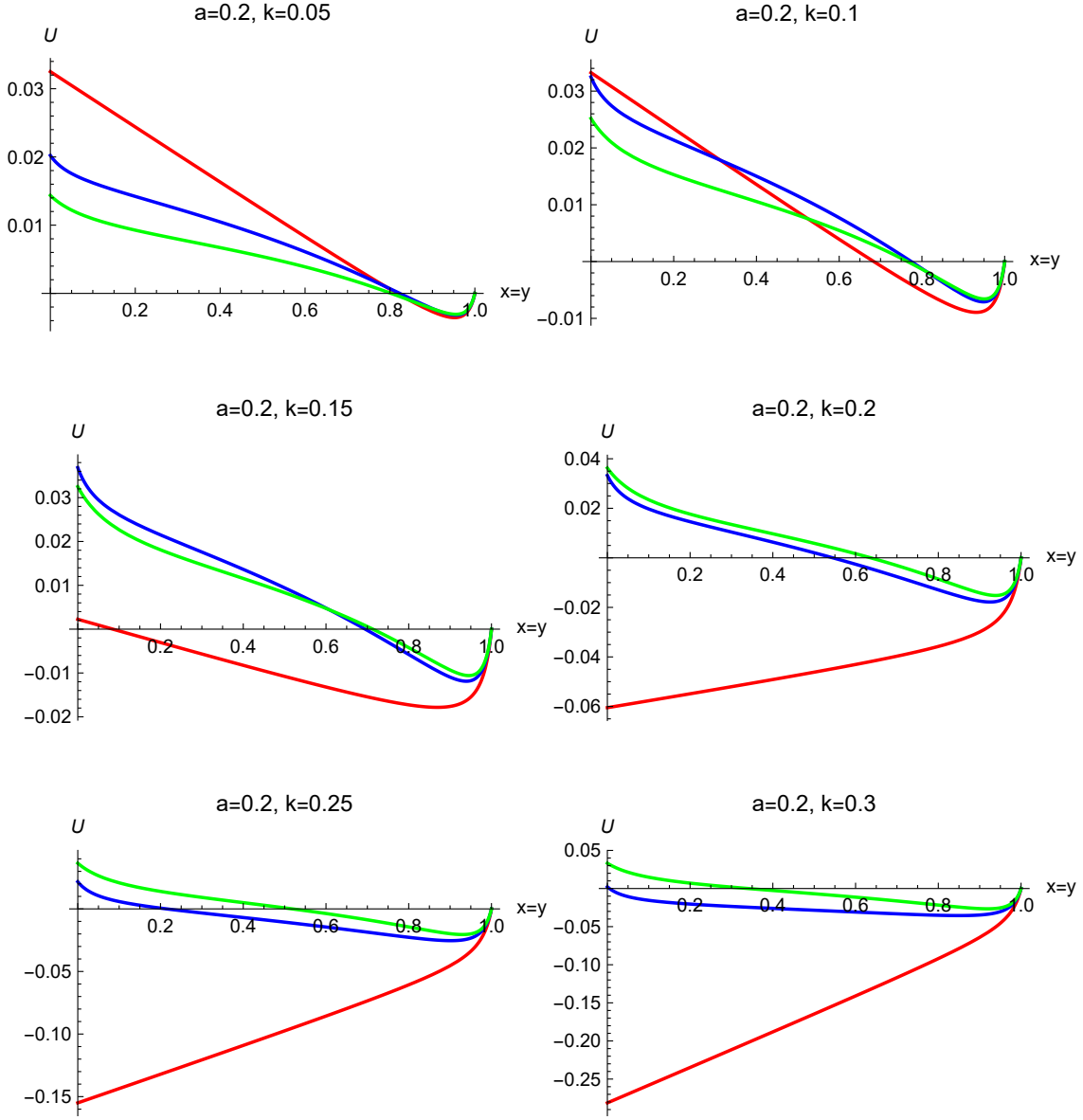


Figure 12: Payoffs under symmetric Markov shocks for divisible (green for  $k/3$  and blue for  $k/2$ ) and indivisible (red) capacities in the same graphs for different  $k$  and  $a = 0.2$ .

#### 4.4 Binding capacity constraints

So far generators are unconstrained in their ability to supply energy; in consequence they always supply according to their best response and clearing prices are standard “Cournot prices” with limited markups that reflect the relative competitiveness of the market. In electricity, binding capacity constraints are a major concern – for then aggregate supply may not meet demand – and they are reflected in widely fluctuating prices. Large price fluctuations invite arbitrage. In this section we study this environment.

We consider the same model with  $n$  generators with large capacity for it to never be a constraint (as before), but also with  $m$  generators of smaller capacity  $\kappa$ . Let  $q_n$  and  $q_m$  be corresponding quantities for each type of the generators. The point can be made by confining ourselves to (a) binary shocks and (b) a storage unit that sells either 0 or  $k$  – as in the introductory example. Then we can adapt the quantities and prices from this example almost directly:

**Lemma 8.** *If the shock is positive ( $\varepsilon = a$ ), then the storage unit is a seller with  $\delta k$  units to sell and the equilibrium price  $p^*$  and equilibrium quantities  $s^*$ ,  $q_n^*$ , and  $q_m^*$  under Cournot competition are:*

$$\begin{aligned} p^* &= \frac{1+a-\delta k}{n+m+1}, & s^* &= \delta k, & q_n^* &= q_m^* = \frac{1+a-\delta k}{n+m+1} & \text{if } \kappa \geq \frac{1+a-k}{n+m+1}; \\ p^* &= \frac{1+a-\delta k-m\kappa}{n+1}, & s^* &= k, & q_n^* &= \frac{1+a-\delta k-m\kappa}{n+1}, & q_m^* &= \kappa \quad \text{if } \kappa \leq \frac{1+a-k}{n+m+1}. \end{aligned}$$

*If the shock is negative ( $\varepsilon = -a$ ), then the storage unit is a buyer with  $k$  units to purchase and the equilibrium price  $p^*$  and equilibrium quantities  $b^*$ ,  $q_n^*$ , and  $q_m^*$  under Cournot competition are:*

$$\begin{aligned} p^* &= \frac{1-a+k}{n+m+1}, & s^* &= k, & q_n^* &= q_m^* = \frac{1-a+k}{n+m+1} & \text{if } \kappa \geq \frac{1-a+k}{n+m+1}; \\ p^* &= \frac{1-a+k-m\kappa}{n+1}, & s^* &= k, & q_n^* &= \frac{1-a+k-m\kappa}{n+1} & q_m^* &= \kappa, \quad \text{if } \kappa \leq \frac{1-a+k}{n+m+1}. \end{aligned}$$

As one expects, price increases and quantities decrease when constraints start binding. As elsewhere in this paper, we compute the charging costs  $B$  (when purchasing) and revenues  $A$  (when selling), and expected payoff, as follows:

**Proposition 9.** *In the presence of capacity constraints, the costs and revenues of a storage*

operator are

$$B = \frac{1 - a + k}{n + m + 1} \cdot k, \quad A = \frac{1 + a - \delta k}{n + m + 1} \cdot \delta k \quad \text{if } \kappa \geq \frac{1 + a - k}{n + m + 1}; \quad (24)$$

$$B = \frac{1 - a + k}{n + m + 1} \cdot k, \quad A = \frac{1 + a - \delta k - m\kappa}{n + 1} \cdot \delta k \quad \text{if } \frac{1 - a + k}{n + m + 1} \leq \kappa \leq \frac{1 + a - k}{n + m + 1}; \quad (25)$$

$$B = \frac{1 - a + k - m\kappa}{n + 1} \cdot k, \quad A = \frac{1 + a - \delta k - m\kappa}{n + 1} \cdot \delta k \quad \text{if } \kappa \leq \frac{1 - a + k}{n + m + 1}. \quad (26)$$

The corresponding payoffs read

$$U_s = \frac{1}{2} \left[ -B + \frac{\beta}{2(1 - \beta)} (A - B) \right].$$

This payoff function is easy to understand: it is the NPV of the arbitrage spread  $A - B$ , accounting for the fact that the storage operator can buy and sell at most every other period and net of the first charge  $B$ . This expression for  $U_s$  is as in (6), modulo the definitions of  $A$  and  $B$ . We can also see that only the intermediate case is interesting: when the capacity constraint binds when charging and discharging, the payoffs are simply re-scaled.<sup>18</sup> Hence we focus on (25). In what follows, let  $\kappa = 1/(n + m + 1)$ .

We would like to understand how changes in the number of generators with limited capacity  $m$  affect storage profits. One can reasonably anticipate that as supply becomes more constrained, prices increase and the payoff to storage increases in consequence. Indeed, let  $N = m + n$  and let  $m$  vary simply from 0 to 10, when the capacity constraint binds,  $q_n$  decreases and  $p^*$  increases. As before, consider  $\delta = \beta = 0.95$ . Also, assume  $n = 4$  and  $k = 0.4$ .

Our first figure (Figure 13) conforms with intuition: fix the number  $N$  but increase  $m$  – so a larger fraction of the generators become constrained – and the payoff to the storage unit uniformly increases. This is simply due to the fact that arbitrage  $A - B$  become increasingly more profitable. As  $m$  keeps increasing, so does this spread and the storage unit becomes the only supplier that can meet demand in the high demand state. This makes for a very large

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<sup>18</sup>Cases (24) and (26) are exactly the same as we considered in the introductory example: we just have  $n + m$  symmetric generators instead of  $n$  ones in case (24) and some shift  $m\kappa$  in demand in case (26). Thus, the only potentially interesting change of behavior may be observed in case (25).

market power, as well as a large spread; the payoff is convex in  $m$  for all values of the shock  $a$  we consider. Indeed, in (25),  $n + m + 1 = N + 1$  is fixed and  $n + 1 = N - m + 1$  keeps decreasing, so  $A - B$  keeps widening.

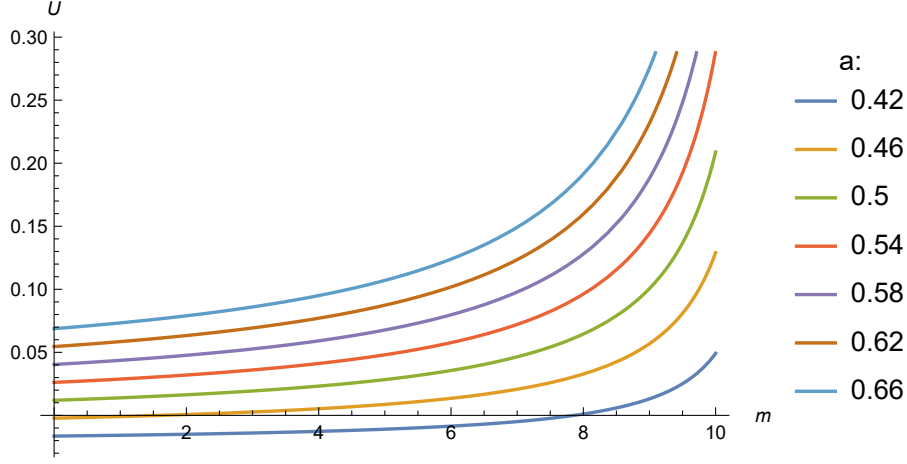


Figure 13: Change of monotonicity in storage payoff functions in case (25) under different  $a$ .

An alternative take on increasing  $m$  is to fix  $n$  but vary  $m$  – so that the total  $N$  also varies. That is, the proportion of constrained generators vary, but the total system capacity and the number of players also increase. We represent this in Figure 14. It shows that the payoffs to storage increase with the number  $m$  of generators with binding capacity when  $a$  is close to  $k$ , but decrease when  $a$  is significantly larger than  $k$ .

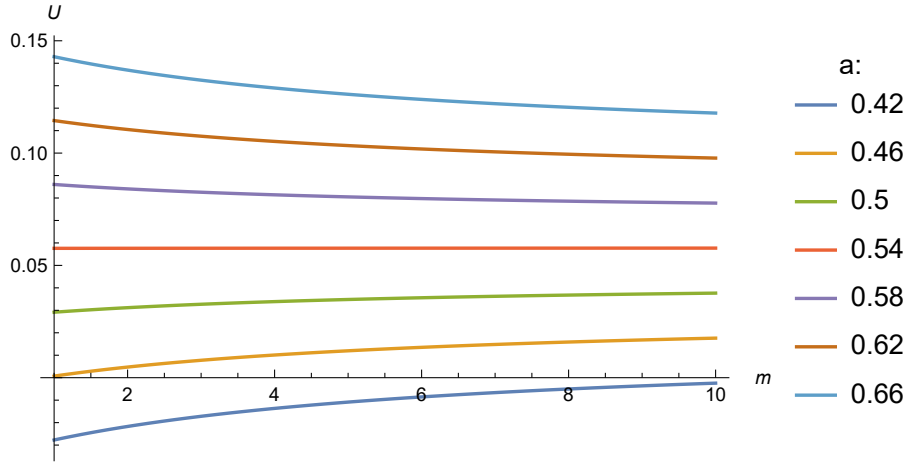


Figure 14: Change of monotonicity in storage payoff functions in case (25) under different  $a$ .

Things are quite different now. This difference stems from the asymmetric effect of storage on prices when buying and selling. The storage operator buys when the shock is negative and *no* generator is constrained, but it sells only when  $m$  of the  $N$  generators are constrained. So

now the combination of  $a$  and  $m$  matters. While increasing  $a$  linearly increases both  $A$  and  $B$ , and therefore the spread  $A - B$ , increasing  $m$  geometrically decreases  $B$  but linearly decreases  $A$ . When  $a$  is large, the geometric effect on  $B$  dominates so increasing  $m$  negatively affects the payoff to storage. The market is “too competitive”. When  $a$  is small, the linear effect on  $A$  dominates. The net effect is that the payoff to storage increases in  $m$ .

## 4.5 Implications for competition policy and market design

Our results reveal a strategic storage operator withholds quantities for two essential reasons. One, they want to exercise their market power; second, they seek to actively manage the continuation risk.

On both accounts it is socially best for the storage units to remain small. It is tautological but nonetheless useful to recall that it is easier to mitigate the exercise of market power if storage has no market power. A small unit (compared to the magnitude of the shocks) uses its full capacity every time it trades. The reason is that the arbitrage gains dominate the continuation risk. Whether any of this is implementable in practice depends in part on the exact storage technology. Economies of scale favour a large size, but batteries tend to display constant returns to scale.

On a different register, [Andres-Cerezo and Fabra \(2023\)](#) show storage and conventional generation should not be integrated, for integration enhances market power. We also show that the storage operator can be excluded by colluding generators. But if the storage operator can charge anyway, or charge at a preferential price, then exclusion may not occur. Hence some qualified integration of storage and generation, for selected generation technologies (e.g. solar), may facilitate the emergence and operation of storage. The difference is that storage is a substitute for thermal generation (at critical times) but it is (typically) a complement for renewable generation. These benefits do not exist in the work of [Andres-Cerezo and Fabra \(2023\)](#), since exclusion is not considered and there is no distinction between generation technologies. This is also not a consideration in [Butters et al. \(Working Paper\)](#), where storage is competitive by assumption.

There may also be a role for a market operator to play insuring the storage unit against the risk of being stuck either full or empty. The operator could tax transactions and disburse these premia to encourage a full unit to sell at a low price (when the shock is negative), or an empty one to buy at a high price (conversely). We note this counters the insurance service implicit in the activities of the storage unit, and increases price volatility. However this is not very different from a reinsurance contract.

Finally, we ask, but are yet unable to answer, how should a bidding space be defined and a market be cleared when bidders use dynamic strategies. Indeed, if storage units are forward-looking, so should be the market operator.

## 5 Conclusion

In this paper we study the dynamic trading of electricity based on storage. This is an important step to understand the economics of electricity storage, and to tackle the ambitious question of market design with storage. We limit ourselves to a Cournot environment with stochastic shocks that follow a Markov chain, and allow for capacity constraints; there is no change in the mean demand. This environment features market power and strategic behaviour, in departure to much of the literature on (other forms of) storage.

Even then, the analysis of such a simple problem is very demanding. To make progress, we must confine ourselves to studying simple heuristics, which allows us to derive explicit forms for the long-horizon payoffs of the storage unit. We are confident these payoffs are close approximations. We uncover two competing forces that a storage operator must balance: market power, which is quite standard, and the continuation risk, which is completely new. That risk is the expected cost to be stuck either empty or full and being unable to either buy or sell, and it is only relevant when storage has market power. Finally, the impact of a capacity constraint on conventional generators has ambiguous effects, depending on how exactly it affects aggregate supply. The full implications of these behaviours on market design are yet to be understood.

There is still a tremendous amount of work to do to really understand the economics of

storage. In this model there is no change in the mean demand over time. This is a central to electricity markets but difficult to incorporate in a model of dynamic trading.



# A Appendix – for online publication

## A.1 Proof of the Introductory example

According to (4) and (2), the storage operator buys  $k$  units under price  $p = (1 - a + k)/(n + 1)$  and sells  $\delta k$  units under price  $p = (1 + a - \delta k)/(n + 1)$ . The probability that the storage unit observes the first positive shock after  $i$  periods is  $(1/2)^{i-1} \cdot (1/2)$ . Thus, the total discounting of waiting for the positive shock after recharging is equal to

$$\frac{1}{2} \cdot \beta + \frac{1}{4} \cdot \beta^2 + \cdots + \frac{1}{2^i} \cdot \beta^i + \cdots = \frac{\beta}{2 - \beta}.$$

Hence, entering the market is profitable for the storage operator if  $-B + \frac{\beta}{2 - \beta}A > 0$ .

Four possible deviations of the storage unit should be considered. All other deviations are just compositions of those four.

- The unit is full but deviates by not selling under the positive shock. Then there may be only loss comparing to the default strategy. Indeed, nothing changes on the market except the future profits to be discounted by  $\beta$ .
- The unit is empty and deviates by not buying under the negative shock. Also, no gains here.
- The unit is full and deviates by selling under the negative shock. In this situation, the quantities supplied by the generators are  $q = (1 - a)/(n + 1)$ . The resulting price after the deviation is

$$p = 1 - a - \delta k - n \frac{1 - a}{n + 1} = \frac{1 - a}{n + 1} - \delta k.$$

To make this deviation profitable, the storage operator must gain more than if it waits for the positive shock and sells in that period:

$$\left( \frac{1 - a}{n + 1} - \delta k \right) \delta k > \frac{\beta}{2 - \beta} \frac{1 + a - \delta k}{n + 1} \delta k,$$

which is impossible when  $B < \frac{\beta}{2 - \beta}A$ .

- The unit is empty and deviates by buying under the positive shock. Here we have  $q = (1 + a)/(n + 1)$ , and the resulting price after the deviation is  $p = (1 + a)/(n + 1) + k$ .

The profits after selling the purchased energy are:

$$\begin{aligned}
& - \left( \frac{1 + a}{n + 1} + k \right) k + \frac{\beta}{2 - \beta} \frac{1 + a - \delta k}{n + 1} \delta k \\
& = - \frac{k}{n + 1} \left( \left( 1 - \frac{\beta}{2 - \beta} \delta \right) (1 + a) + \left( \frac{\beta}{2 - \beta} \delta^2 + n + 1 \right) k \right) < 0.
\end{aligned}$$

There are no gains from this deviation.

Ruling out deviations of the generators is simple. In each round, we have a static Cournot equilibrium for all the participants. Thus, any change of the equilibrium quantity in round  $t$  leads to decreasing the payoffs in that round and, thus, decreasing the overall payoffs. Indeed, the stage-game Cournot equilibrium is an equilibrium also in the long-horizon game.

To find the expected payoff of the storage unit, let's introduce value function  $V_t(i)$ ,  $i \in \{1, 0\}$ .  $V_t(i)$  is the total expected payoff of the unit from moment  $t$  if the current state is full ( $i = 1$ ) or empty ( $i = 0$ ). We have a system of recursive equations:

$$\begin{cases} V_t(1) = \frac{1}{2} \cdot (A + \beta V_{t+1}(0)) + \frac{1}{2} \cdot \beta \cdot V_{t+1}(1), \\ V_t(0) = \frac{1}{2} \cdot \beta \cdot V_{t+1}(0) + \frac{1}{2} \cdot (-B + \beta \cdot V_{t+1}(1)). \end{cases}$$

It can be rewritten in a matrix form

$$V_t = P + \beta \cdot Q \cdot V_{t+1}, \quad (27)$$

where

$$V_t = \begin{pmatrix} V_t(1) \\ V_t(0) \end{pmatrix}, \quad P = \frac{1}{2} \begin{pmatrix} A \\ -B \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that  $Q^2 = Q$ . For  $\beta < 1$ , we can find from (27) that

$$\begin{aligned}
V_0 &= P + \sum_{i=1}^t \beta^i Q^i \cdot P + \beta^{t+1} Q^{t+1} \cdot V_{t+1} = \\
&= P + \frac{\beta(1 - \beta^t)}{1 - \beta} \cdot Q \cdot P + \beta^{t+1} \cdot Q \cdot V_{t+1} \xrightarrow{t \rightarrow \infty} P + \frac{\beta}{1 - \beta} \cdot Q \cdot P = \frac{1}{4} \begin{pmatrix} \frac{2-\beta}{1-\beta}A - \frac{\beta}{1-\beta}B \\ \frac{\beta}{1-\beta}A - \frac{2-\beta}{1-\beta}B \end{pmatrix}.
\end{aligned} \tag{28}$$

The lower term is exactly  $U_s$ .

To find the expected payoff of the generators, let's introduce value function  $W_t(i)$ ,  $i \in \{1, 0\}$ .  $W_t(i)$  is the total expected payoff of a generator from moment  $t$  if the current state of the storage unit is full ( $i = 1$ ) or empty ( $i = 0$ ). We have a system of recursive equations:

$$\begin{cases} W_t(1) = \frac{1}{2} \cdot (G_{10} + \beta W_{t+1}(1)) + \frac{1}{2} \cdot (G_{11} + \beta \cdot W_{t+1}(0)), \\ W_t(0) = \frac{1}{2} \cdot (G_{00} + \beta \cdot W_{t+1}(1)) + \frac{1}{2} \cdot (G_{01} + \beta \cdot W_{t+1}(0)). \end{cases}$$

It can be rewritten in a matrix form

$$W_t = R + \beta \cdot Q \cdot W_{t+1},$$

where

$$W_t = \begin{pmatrix} W_t(1) \\ W_t(0) \end{pmatrix}, \quad R = \frac{1}{2} \begin{pmatrix} G_{10} + G_{11} \\ G_{00} + G_{01} \end{pmatrix}, \quad Q = Q^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Using the same algebra as for  $V(t)$  earlier, we obtain

$$W_0 \xrightarrow{t \rightarrow \infty} R + \frac{\beta}{1 - \beta} \cdot Q \cdot R = \begin{pmatrix} \frac{1}{2}(G_{10} + G_{11}) + \frac{\beta}{4(1-\beta)}(G_{10} + G_{11} + G_{00} + G_{01}) \\ \frac{1}{2}(G_{00} + G_{01}) + \frac{\beta}{4(1-\beta)}(G_{10} + G_{11} + G_{00} + G_{01}) \end{pmatrix}.$$

The lower term is exactly  $U_g$ .

To find cumulative consumers' expected payments per period, notice first that each of four states – a combination of a full/empty storage unit and a positive/negative shock – has the same probability  $1/4$ . Table 1 summarizes all the consumers' payments under each of those events.

shock \ storage	empty	full
negative	$\frac{1-a+k}{n+1} \cdot \left( n \frac{1-a+k}{n+1} - k \right)$	$\frac{1-a}{n+1} \cdot n \frac{1-a}{n+1}$
positive	$\frac{1+a}{n+1} \cdot n \frac{1+a}{n+1}$	$\frac{1+a-k}{n+1} \cdot \left( n \frac{1+a-k}{n+1} + k \right)$

Table 1

Summing up all the cells with weight  $1/4$  for each, we get  $C_1$ . ■

## A.2 Proofs of the Propositions

*Proof of Proposition 3.* Since under proportional bids, we never reach upper limit  $k$  and never reach lower limit 0 again after starting there, the system of equations (10) takes the following form:

$$\begin{cases} V(0) &= \frac{1}{2-\beta} \left( -\frac{1-a+rk}{n+1} \cdot rk + \beta V(rk) \right), \\ V(c) &= \frac{1}{2} \left( -\frac{1-a+r(k-c)}{n+1} \cdot r(k-c) + \beta V(c+r(k-c)) \right) \\ &\quad + \frac{1}{2} \left( \frac{1+a-\delta rc}{n+1} \cdot \delta rc + \beta V((1-r)c) \right). \end{cases}$$

We need to find  $V(0) = U_s^P$ . Let's enumerate all  $c$ ,  $b(c)$ , and  $a(c)$  in order of their appearance when we expand our equation for  $V(0)$ . Namely, in the first period we have

$$V(0) = \frac{1}{2} \beta V(0) + \frac{1}{2} \left( -\frac{1-a+rk}{n+1} \cdot rk + \beta V(rk) \right) = -\frac{1}{2} B(rk) + \frac{\beta}{2} (V(0) + V(rk)),$$

so we define  $c^0 = 0$ ,  $b^0 = r(k - c^0) = rk$ ,  $c^1 = c^0 + b^0 = rk$ . In the second round, we get

$$\begin{aligned}
V(0) &= -\frac{1}{2}B(rk) + \frac{\beta}{2} \left( \frac{1}{2}\beta V(0) + \frac{1}{2}(-B(rk) + \beta V(rk)) \right. \\
&\quad \left. + \frac{1}{2}(-B(r(k - rk)) + \beta V(rk + r(k - rk))) + \frac{1}{2}(A(r \cdot rk) + \beta V((1 - r)rk)) \right) \\
&= -\frac{1}{2}B(rk) + \frac{\beta}{4} \left( -B(rk) - B(r(1 - r)k) + A(r^2k) \right) \\
&\quad + \frac{\beta^2}{4} \left( V(0) + V(rk) + V(r(2 - r)k) + V(r(1 - r)k) \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
b^1 &= r(k - c^1) = r(1 - r)k, & a^1 &= rc^1 = r^2k, \\
c^2 &= c^1 + b^1 = r(2 - r)k, & c^3 &= c^1 - a^1 = r(1 - r)k.
\end{aligned}$$

In the third round, we get

$$\begin{aligned}
V(0) &= -\frac{1}{2}B(b^0) + \frac{\beta}{4}(-B(b^0) - B(b^1) + A(a^1)) \\
&\quad + \frac{\beta^2}{8}(-B(b^0) - B(b^1) - B(b^2) - B(b^3) + A(a^1) + A(a^2) + A(a^3)) + \frac{\beta^3}{8} \sum_{i=0}^7 V(c^i),
\end{aligned}$$

where

$$\begin{aligned}
b^2 &= r(k - c^2) = r(1 - 2r - r^2)k, & b^3 &= r(k - c^3) = r(1 - r + r^2)k, \\
a^2 &= rc^2 = r^2(2 - r)k, & a^3 &= rc^3 = r^2(1 - r)k, \\
c^4 &= c^2 + b^2 = r(3 - 3r - r^2)k, & c^5 &= c^3 + b^3 = r(2 - 2r + r^2)k, \\
c^6 &= c^2 - a^2 = r(1 - r)(2 - r)k, & c^7 &= c^3 - a^3 = r(1 - r)^2k.
\end{aligned}$$

Finally, in round  $t$ ,

$$\begin{aligned} V(0) = & -\frac{1}{2}B(b^0) + \frac{\beta}{4}(-B(b^0) - B(b^1) + A(a^1)) + \frac{\beta^2}{8}\left(-\sum_{i=0}^3 B(b^i) + \sum_{i=1}^3 A(a^i)\right) + \dots \\ & + \frac{\beta^{t-1}}{2^t}\left(-\sum_{i=0}^{2^{t-1}-1} B(b^i) + \sum_{i=1}^{2^{t-1}-1} A(a^i)\right) + \frac{\beta^t}{2^t}\sum_{i=0}^{2^t-1} V(c^i), \end{aligned}$$

where  $b^i$ ,  $a^i$ , and  $c^i$  can be found recursively. Continuing this process infinitely and noticing that

$$\frac{\beta^t}{2^t}\sum_{i=0}^{2^t-1} V(c^i) \leq \beta^t V(\max_i c^i) \xrightarrow[t \rightarrow \infty]{} 0,$$

we obtain:

$$V(0) \xrightarrow[t \rightarrow \infty]{} -\frac{1}{2}B(b^0) + \frac{1}{2}\sum_{j=1}^{\infty} \left(\frac{\beta}{2}\right)^j \left(-\sum_{i=0}^{2^j-1} B(b^i) + \sum_{i=1}^{2^j-1} A(a^i)\right). \quad (29)$$

Let

$$G(t) = \sum_{i=0}^{2^t-1} c^i, \quad H(t) = \sum_{i=0}^{2^t-1} (c^i)^2.$$

We can get a recursive equation for  $G(t)$ :

$$\begin{aligned} G(t) &= \sum_{i=0}^{2^{t-1}-1} (c^i + b^i) + \sum_{i=1}^{2^{t-1}-1} (c^i - a^i) \\ &= \sum_{i=0}^{2^{t-1}-1} (c^i + r(k - c^i)) + \sum_{i=0}^{2^{t-1}-1} (c^i - rc^i) = rk2^{t-1} + 2(1-r)G(t-1), \end{aligned}$$

which implies

$$\begin{aligned} G(t) &= rk2^{t-1} + 2(1-r)(rk2^{t-2} + 2(1-r)G(t-2)) = \dots \\ &= rk \sum_{i=0}^{t-1} 2^i (1-r)^i 2^{t-1-i} + 2^t (1-r)^t G(0) = k2^{t-1} (1 - (1-r)^t). \end{aligned} \quad (30)$$

The same technique works for  $H(t)$ :

$$\begin{aligned}
H(t) &= \sum_{i=0}^{2^{t-1}-1} (c^i + b^i)^2 + \sum_{i=1}^{2^{t-1}-1} (c^i - a^i)^2 = \sum_{i=0}^{2^{t-1}-1} (rk + (1-r)c^i)^2 + \sum_{i=0}^{2^{t-1}-1} ((1-r)c^i)^2 \\
&= r^2 k^2 2^{t-1} + 2r(1-r)kG(t-1) + (1-r)^2 H(t-1) + (1-r)^2 H(t-1) \\
&= rk^2 2^{t-1} (1 - (1-r)^t) + 2(1-r)^2 H(t-1),
\end{aligned}$$

and

$$\begin{aligned}
H(t) &= rk^2 2^{t-1} (1 - (1-r)^t) + 2(1-r)^2 \left( rk^2 2^{t-2} (1 - (1-r)^{t-1}) + 2(1-r)^2 H(t-2) \right) \\
&= \dots = rk^2 \sum_{i=0}^{t-1} 2^i (1-r)^{2i} 2^{t-i-1} (1 - (1-r)^{t-i}) + 2^t (1-r)^{2t} H(0) \\
&= k^2 2^{t-1} \frac{(1 - (1-r)^t) (1 - (1-r)^{t+1})}{2-r}.
\end{aligned} \tag{31}$$

From (30), we can find expressions for sums of  $b^i$  and  $a^i$ :

$$\begin{aligned}
\sum_{i=0}^{2^t-1} b^i &= \sum_{i=0}^{2^t-1} r(k - c^i) = rk2^t - rG(t) = rk2^{t-1} (1 + (1-r)^t), \\
\sum_{i=0}^{2^t-1} a^i &= \sum_{i=0}^{2^t-1} rc^i = rG(t) = rk2^{t-1} (1 - (1-r)^t).
\end{aligned}$$

From (30) and (31), we can find expressions for sums of squares of  $b^i$  and  $a^i$ :

$$\begin{aligned}
\sum_{i=0}^{2^t-1} (b^i)^2 &= \sum_{i=0}^{2^t-1} r^2 (k - c^i)^2 \\
&= r^2 k^2 2^t - 2r^2 kG(t) + r^2 H(t) = r^2 k^2 2^{t-1} \left( \frac{1 + (1-r)^{2t+1}}{2-r} + (1-r)^t \right), \\
\sum_{i=0}^{2^t-1} (a^i)^2 &= \sum_{i=0}^{2^t-1} r^2 (c^i)^2 = r^2 H(t) = r^2 k^2 2^{t-1} \left( \frac{1 + (1-r)^{2t+1}}{2-r} - (1-r)^t \right).
\end{aligned}$$

Now we are ready to calculate sums of  $B(b(i))$  and  $A(a(i))$  and get the final formula for

$V(0)$ . From (29), we have

$$\begin{aligned}
V(0) &= -\frac{1}{2}B(rk) + \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{\beta}{2}\right)^j \left( -\sum_{i=0}^{2^j-1} \frac{1-a+b^i}{n+1} b^i + \sum_{i=1}^{2^j-1} \frac{1+a-\delta a^i}{n+1} \delta a^i \right) \\
&= -\frac{1}{2} \frac{1-a+rk}{n+1} rk + \frac{1}{2(n+1)} \sum_{j=1}^{\infty} \left(\frac{\beta}{2}\right)^j \left( -(1-a)rk2^{j-1} (1+(1-r)^j) \right. \\
&\quad \left. - r^2 k^2 2^{j-1} \left( \frac{1+(1-r)^{2j+1}}{2-r} + (1-r)^j \right) + (1+a)\delta rk 2^{j-1} (1-(1-r)^j) \right. \\
&\quad \left. - \delta^2 r^2 k^2 2^{j-1} \left( \frac{1+(1-r)^{2j+1}}{2-r} - (1-r)^j \right) \right) \\
&= \frac{rk}{4(n+1)(1-\beta)(1-(1-r)\beta)} \left( r\beta((1+a)\delta - (1-a)) - 2(1-\beta)(1-a+rk) \right. \\
&\quad \left. - rk \frac{\beta r^2(1+\delta^2)}{1-(1-r)^2\beta} \right).
\end{aligned}$$

The last expression is exactly formula (11).  $\square$

*Proof of Proposition 4.* The system of equations (10) takes the following form:

$$\begin{cases} V_t\left(\frac{ik}{m}\right) = \frac{1}{2} \left( A\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{(i-1)k}{m}\right) - B\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{(i+1)k}{m}\right) \right), \\ V_t(0) = \frac{1}{2} \left( \beta V_{t+1}(0) - B\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{k}{m}\right) \right), \\ V_t(k) = \frac{1}{2} \left( A\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{(m-1)k}{m}\right) + \beta V_{t+1}(k) \right) \end{cases} \quad (32)$$

for all  $1 \leq i \leq m-1$ . We are interested in coefficients  $c_t^i$  in front of value functions  $V_t(ik/m)$  for each particular  $t \geq 0$  and  $0 \leq i \leq m$ , such that

$$V_0(0) = F_{t-1} \left( \beta, A\left(\frac{k}{m}\right), B\left(\frac{k}{m}\right) \right) + \left(\frac{\beta}{2}\right)^t \sum_{i=0}^m c_t^i V_t\left(\frac{ik}{m}\right). \quad (33)$$

Note that  $\sum_i c_t^i = 2^t$  for any  $t$ .

In each period  $t$ , the storage unit buys energy with probability  $1/2$ . This cannot be done only if the unit has reached its full capacity. Also, in period  $t$  the storage unit sells energy with probability  $1/2$  if it's not empty. Thus, the overall expected earnings of storage up to



period  $t$  can be described by the following expression:

$$F_t = \sum_{i=0}^t \left(\frac{\beta}{2}\right)^i \cdot \frac{1}{2} \cdot \left( (2^i - c_i^0)A\left(\frac{k}{m}\right) - (2^i - c_i^m)B\left(\frac{k}{m}\right) \right).$$

Indeed,  $c_t^0/2^t$  and  $c_t^m/2^t$  are the probabilities of the storage to be correspondingly empty or full at  $t$ , according to (33).

Since  $\sum_i c_t^i = 2^t$ ,  $\beta < 1$ , and  $V_t$  is a nondecreasing function, the last term in (33) goes to zero if  $t \rightarrow \infty$ :

$$\left(\frac{\beta}{2}\right)^t \sum_{i=0}^m c_t^i V_t\left(\frac{ik}{m}\right) \leq \left(\frac{\beta}{2}\right)^t \sum_{i=0}^m c_t^i V_t(k) = \beta^t V_t(k) \xrightarrow{t \rightarrow \infty} 0.$$

Then the expected payoff function takes the following form:

$$U_s^C = V_0(0) = \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^i \cdot \frac{1}{2} \cdot \left( (2^i - c_i^0)A\left(\frac{k}{m}\right) - (2^i - c_i^m)B\left(\frac{k}{m}\right) \right). \quad (34)$$

We need to find  $c_i^0$  and  $c_i^m$ . From (32), we can see that

$$c_{t+1}^0 = c_t^0 + c_t^1, \quad c_{t+1}^i = c_t^{i-1} + c_t^{i+1} \quad (1 \leq i \leq m-1), \quad c_{t+1}^m = c_t^{m-1} + c_t^m.$$

Thus, we have a modified version of Pascal's triangle. Let's see if we can express  $c_t^i$  in terms of binomial coefficients  $C_n^k = n!/(k!(n-k)!)$ . The two main properties of  $C_n^k$  we are going to use are

$$C_n^k = C_n^{m-k}, \quad C_n^k + C_n^{k+1} = C_{n+1}^{k+1}.$$

We start with  $c_0^0 = 1 = C_0^0$  (and all other  $c_0^i = 0$ ,  $i \geq 1$ ). In period 1, we have  $c_1^1 = c_1^0 = 1 = C_1^0$  with all other  $c_1^i$  equal to zero. Period 2 delivers  $c_2^2 = c_2^1 = 1 = C_2^0$  and  $c_2^0 = c_1^1 + c_1^0 = C_1^0 + C_1^0 = C_1^1 + C_1^0 = C_2^1$ , with all remaining  $c_1^i$  equal to zero. In period 3, we have  $c_3^3 = c_3^2 = 1 = C_3^0$ ,  $c_3^1 = c_2^2 + c_2^0 = C_2^0 + C_2^1 = C_3^1$ , and  $c_3^0 = c_2^1 + c_2^0 = C_2^0 + C_2^1 = C_3^1$ , with all other  $c_1^i$  equal to zero. We get rid of all the zeroes by period  $m$  with  $c_m^i = C_m^{\lfloor (m-i)/2 \rfloor}$

(here,  $\lfloor x \rfloor$  is the largest integer which is less than or equal to  $x$ ). This process is summarized in Table 2.

state of charge $i$	$m$	0	0	.	.	.	.	.	.	0	0	$C_m^0$
	$m-1$	0	0	.	.	.	.	.	.	0	$C_{m-1}^0$	$C_m^0$
	$m-2$	0	0	.	.	.	.	.	.	$C_{m-2}^0$	$C_{m-1}^0$	$C_m^1$
	$\dots$	$\vdots$							$\vdots$			$\vdots$
	2	0	0	$C_2^0$	$C_3^0$	$C_4^1$	.	.	.	$C_{m-2}^{\lfloor (m-4)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-3)/2 \rfloor}$	$C_m^{\lfloor (m-2)/2 \rfloor}$
	1	0	$C_1^0$	$C_2^0$	$C_3^1$	$C_4^1$	.	.	.	$C_{m-2}^{\lfloor (m-3)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-2)/2 \rfloor}$	$C_m^{\lfloor (m-1)/2 \rfloor}$
0		$C_0^0$	$C_1^0$	$C_2^1$	$C_3^1$	$C_4^2$	.	.	.	$C_{m-2}^{\lfloor (m-2)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-1)/2 \rfloor}$	$C_m^{\lfloor m/2 \rfloor}$
		0	1	2	3	4	.	.	.	$m-2$	$m-1$	$m$
$c_t^i$		time $t$										

Table 2: The first  $m+1$  steps of evolving  $c_t^i$

However, after period  $m$  we cannot go up anymore. Instead, all the extra mass we accumulate goes down step by step. Namely, in period  $m+1$  we still have  $c_{m+1}^i = C_{m+1}^{\lfloor (m+1-i)/2 \rfloor}$  for all  $0 \leq i \leq m-1$ , but for  $i = m$  we now have  $c_{m+1}^m = C_{m+1}^0 + C_{m+1}^0$ . In period  $m+2$ , we still have  $c_{m+2}^i = C_{m+2}^{\lfloor (m+2-i)/2 \rfloor}$ , but only for  $0 \leq i \leq m-2$ . For  $i = m$  and  $i = m-1$ , we have  $c_{m+2}^m = c_{m+2}^{m-1} = C_{m+2}^1 + C_{m+2}^0$ , etc. Finally, in period  $2m+1$ , we have

$$c_{2m+1}^i = C_{2m+1}^{\lfloor \frac{2m+1-i}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{i}{2} \rfloor} \quad 0 \leq i \leq m.$$

See Table 3 for the entire picture.

<b>m</b>	$C_m^0$	$C_{m+1}^0 + C_{m+1}^0$	$C_{m+2}^1 + C_{m+2}^0$	$\dots$	$C_{2m-1}^{\lfloor \frac{m-1}{2} \rfloor} + C_{2m-1}^{\lfloor \frac{m-2}{2} \rfloor}$	$C_{2m}^{\lfloor \frac{m}{2} \rfloor} + C_{2m}^{\lfloor \frac{m-1}{2} \rfloor}$	$C_{2m+1}^{\lfloor \frac{m+1}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{m}{2} \rfloor}$
<b>m-1</b>	$C_m^0$	$C_m^1$	$C_{m+2}^1 + C_{m+2}^0$	$\dots$	$C_{2m-1}^{\lfloor \frac{m}{2} \rfloor} + C_{2m-1}^{\lfloor \frac{m-3}{2} \rfloor}$	$C_{2m}^{\lfloor \frac{m+1}{2} \rfloor} + C_{2m}^{\lfloor \frac{m-2}{2} \rfloor}$	$C_{2m+1}^{\lfloor \frac{m+2}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{m-1}{2} \rfloor}$
<b>m-2</b>	$C_m^1$	$C_{m+1}^1$	$C_{m+2}^2$	$\dots$	$C_{2m-1}^{\lfloor \frac{m+1}{2} \rfloor} + C_{2m-1}^{\lfloor \frac{m-4}{2} \rfloor}$	$C_{2m}^{\lfloor \frac{m+2}{2} \rfloor} + C_{2m}^{\lfloor \frac{m-3}{2} \rfloor}$	$C_{2m+1}^{\lfloor \frac{m+3}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{m-2}{2} \rfloor}$
$\dots$	$\vdots$			$\vdots$			$\vdots$
<b>2</b>	$C_m^{\lfloor \frac{m-2}{2} \rfloor}$	$C_{m+1}^{\lfloor \frac{m-1}{2} \rfloor}$	$C_{m+2}^{\lfloor \frac{m}{2} \rfloor}$	$\dots$	$C_{2m-1}^{m-2} + C_{2m-1}^0$	$C_{2m}^{m-1} + C_{2m}^0$	$C_{2m+1}^{m-1} + C_{2m+1}^1$
<b>1</b>	$C_m^{\lfloor \frac{m-1}{2} \rfloor}$	$C_{m+1}^{\lfloor \frac{m}{2} \rfloor}$	$C_{m+2}^{\lfloor \frac{m+1}{2} \rfloor}$	$\dots$	$C_{2m-1}^{m-1}$	$C_{2m}^{m-1} + C_{2m}^0$	$C_{2m+1}^m + C_{2m+1}^0$
<b>0</b>	$C_m^{\lfloor \frac{m}{2} \rfloor}$	$C_{m+1}^{\lfloor \frac{m+1}{2} \rfloor}$	$C_{m+2}^{\lfloor \frac{m+2}{2} \rfloor}$	$\dots$	$C_{2m-1}^{m-1}$	$C_{2m}^m$	$C_{2m+1}^m + C_{2m+1}^0$
<b>i/t</b>	<b>m</b>	<b>m+1</b>	<b>m+2</b>	$\dots$	<b>2m-1</b>	<b>2m</b>	<b>2m+1</b>

Table 3: The second  $m+1$  steps of evolving  $c_t^i$

Now the excess mass reached the lower boundary again. Since we cannot go down anymore, this mass has to spread up again and add one more binomial coefficient as a summand to all  $c_t^i$  starting from  $t = 2m + 2$  and until  $t = 3m + 2$ . This process continues infinitely. We can

now derive  $c_i^0$  and  $c_i^m$ :

$$c_i^0 = \begin{cases} C_i^{\lfloor \frac{i}{2} \rfloor} & \text{if } 0 \leq i \leq 2m, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} & \text{if } i = 2m+1, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} & \text{if } 2m+2 \leq i \leq 2m+2+2m, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1-2(m+1)}{2} \rfloor} & \text{if } i = 2(m+1) + 2m+1, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1-2(m+1)}{2} \rfloor} + C_i^{\lfloor \frac{i-4(m+1)-2m}{2} \rfloor} & \text{if } 4(m+1) \leq i \leq 4(m+1) + 2m, \\ \dots & \end{cases}$$

$$c_i^m = \begin{cases} 0 & \text{if } 0 \leq i \leq m-1, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} & \text{if } i = m, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1}{2} \rfloor} & \text{if } m+1 \leq i \leq m+1+2m, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1-2m-1}{2} \rfloor} & \text{if } i = m+1+2m+1, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-3(m+1)}{2} \rfloor} & \text{if } 3(m+1) \leq i \leq 3(m+1) + 2m, \\ \dots & \end{cases}$$

The coefficient in front of  $A(k/m)$  in (34) takes the following form:

$$\begin{aligned} \frac{1}{2} \cdot \sum_{i=0}^{\infty} \left( \frac{\beta}{2} \right)^i (2^i - c_i^0) &= \frac{1}{2} \cdot \sum_{i=0}^{\infty} \beta^i - \sum_{i=0}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i}{2} \rfloor} - \sum_{i=2m+1}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} - \\ &- \sum_{i=2m+2}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} - \sum_{i=4m+3}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-4m-3}{2} \rfloor} - \dots = \frac{1}{2(1-\beta)} - \sum_{i=0}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i}{2} \rfloor} - \\ &- \sum_{j=0}^{\infty} \left( \sum_{i=2m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-1-2j(m+1)}{2} \rfloor} + \sum_{i=2m+2+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-2-2j(m+1)}{2} \rfloor} \right). \end{aligned}$$

We can rewrite the second term:

$$\sum_{i=0}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i}{2} \rfloor} = \sum_{i=0}^{\infty} \frac{\beta^{2i}}{2^{2i+1}} C_{2i}^i + \sum_{i=0}^{\infty} \frac{\beta^{2i+1}}{2^{2i+2}} C_{2i+1}^i = \frac{1}{2} \cdot \sum_{i=0}^{\infty} \left( \frac{\beta}{2} \right)^{2i} \left( C_{2i}^i + \frac{\beta}{2} C_{2i+1}^i \right).$$

The term in parenthesis can also be simplified by considering even and odd indices separately:

$$\begin{aligned} & \sum_{i=2m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-1-2j(m+1)}{2} \rfloor} + \sum_{i=2m+2+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-2-2j(m+1)}{2} \rfloor} = \\ & = \frac{\beta^{2(j+1)(m+1)-1}}{2^{2(j+1)(m+1)}} + \sum_{i=2(j+1)(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} \left( C_i^{\lfloor \frac{i+1-2(j+1)(m+1)}{2} \rfloor} + C_i^{\lfloor \frac{i-2(j+1)(m+1)}{2} \rfloor} \right) = \\ & = \frac{\beta^{2(j+1)(m+1)-1}}{2^{2(j+1)(m+1)}} + \sum_{i=0}^{\infty} \frac{\beta^{2(j+1)(m+1)+2i}}{2^{2(j+1)(m+1)+2i}} C_{2(j+1)(m+1)+2i}^i + \\ & + \sum_{i=0}^{\infty} \frac{\beta^{2(j+1)(m+1)+2i+1}}{2^{2(j+1)(m+1)+2i+2}} C_{2(j+1)(m+1)+2i+2}^{i+1} = \frac{1+\beta}{\beta} \sum_{i=0}^{\infty} \left( \frac{\beta}{2} \right)^{2(j+1)(m+1)+2i} C_{2(j+1)(m+1)+2i}^i. \end{aligned}$$

The coefficient in front of  $B(k/m)$  in (34) takes the following form:

$$\begin{aligned} -\frac{1}{2} \cdot \sum_{i=0}^{\infty} \left( \frac{\beta}{2} \right)^i (2^i - c_i^m) &= -\frac{1}{2} \cdot \sum_{i=0}^{\infty} \beta^i + \sum_{i=m}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m}{2} \rfloor} + \sum_{i=m+1}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-1}{2} \rfloor} \\ &+ \sum_{i=3m+2}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-3m-2}{2} \rfloor} + \sum_{i=3m+3}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-3m-3}{2} \rfloor} + \dots \\ &= -\frac{1}{2(1-\beta)} + \sum_{j=0}^{\infty} \left( \sum_{i=m+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)}{2} \rfloor} \right. \\ &+ \left. \sum_{i=m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)-1}{2} \rfloor} \right). \end{aligned}$$

The term in parenthesis can be simplified the same way as for  $A(k/m)$ :

$$\begin{aligned} & \sum_{i=m+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)}{2} \rfloor} + \sum_{i=m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)-1}{2} \rfloor} = \\ & = \frac{1+\beta}{\beta} \sum_{i=0}^{\infty} \left( \frac{\beta}{2} \right)^{(2j+1)(m+1)+2i} C_{(2j+1)(m+1)+2i}^i. \end{aligned}$$

Summing everything up, we obtain the overall expected profit of the storage unit from

(34):

$$U_s^C = \frac{A(k/m) - B(k/m)}{2(1 - \beta)} + \frac{1 + \beta}{\beta} \left( B\left(\frac{k}{m}\right) \sum_{j=0}^{\infty} \left(\frac{\beta}{2}\right)^{(m+1)(2j+1)} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} C_{2i+(m+1)(2j+1)}^i - \right. \\ \left. - A\left(\frac{k}{m}\right) \sum_{j=1}^{\infty} \left(\frac{\beta}{2}\right)^{2(m+1)j} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} C_{2i+2(m+1)j}^i \right) - \frac{A(k/m)}{2} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} \left( C_{2i}^i + \frac{\beta}{2} C_{2i+1}^i \right).$$

Using formula

$$\sum_{i=0}^{\infty} \beta^i C_{2i+r}^i = \frac{2^r}{\sqrt{1 - 4\beta} \left(1 + \sqrt{1 - 4\beta}\right)^r}$$

from [Graham et al. \(1994\)](#) (p. 203) and introducing new discounting coefficient

$$\tilde{\beta} = \frac{\beta}{1 + \sqrt{1 - \beta^2}},$$

we finally get

$$U_s^C = \frac{1}{2(1 - \beta)} \left( -B\left(\frac{k}{m}\right) + \tilde{\beta} A\left(\frac{k}{m}\right) - \right. \\ \left. - \frac{2\sqrt{1 - \beta^2}}{\beta} \frac{\tilde{\beta}^{m+1}}{1 - \tilde{\beta}^{2(m+1)}} \left( -B\left(\frac{k}{m}\right) + \tilde{\beta}^{m+1} A\left(\frac{k}{m}\right) \right) \right). \quad (35)$$

□

*Proof of Proposition 5.* Let's prove formula 14 first. In this case of four possible states of charge, the system of equations (10) takes the following form:

$$\left\{ \begin{array}{l} V_t(0) = \frac{1}{2} (\beta V_{t+1}(0) - B(X) + \beta V_{t+1}(X)), \\ V_t(X) = \frac{1}{2} (A(X) + \beta V_{t+1}(0) - B(k - X) + \beta V_{t+1}(k)), \\ V_t(k - X) = \frac{1}{2} (A(k - X) + \beta V_{t+1}(0) - B(X) + \beta V_{t+1}(k)), \\ V_t(k) = \frac{1}{2} (A(X) + \beta V_{t+1}(k - X) + \beta V_{t+1}(k)). \end{array} \right. \quad (36)$$

It can be rewritten in a matrix form

$$V_t = P + \beta \cdot Q \cdot V_{t+1}, \quad (37)$$

where

$$V_t = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -B(X)/2 \\ A(X)/2 - B(k-X)/2 \\ A(k-X)/2 - B(X)/2 \\ A(X)/2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

To calculate power  $t$  of matrix  $Q$ , we find the Jordan decomposition  $Q = T \cdot J \cdot T^{-1}$  of  $Q$ .

Here,

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -2 & 0 & 1 \\ 1 & -2 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

so  $Q^t = T \cdot J^t \cdot T^{-1}$ .

For  $\beta < 1$ , we can find from (37) that

$$\begin{aligned} V_0 &= \sum_{i=0}^t \beta^i Q^i \cdot P + \beta^{t+1} Q^{t+1} \cdot V_{t+1} \xrightarrow{t \rightarrow \infty} \sum_{i=0}^{\infty} \beta^i Q^i \cdot P = \\ &= P + T \cdot \begin{pmatrix} \frac{\beta}{1-\beta} & 0 & 0 & 0 \\ 0 & -\frac{\beta}{2+\beta} & 0 & 0 \\ 0 & 0 & \frac{\beta}{2-\beta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot T^{-1} \cdot P, \end{aligned}$$

from where we finally get  $U_s^X = V_0(0)$ :

$$U_s^X = -\frac{1}{2} \left( B(X) + \frac{\beta}{2-\beta} B(k-X) \right) + \\ + \frac{\beta}{4(1-\beta)} \left( A(X) - B(X) + \frac{\beta^2}{4-\beta^2} (A(k-X) - B(k-X)) \right),$$

which is exactly (14).

Formulae (15) – (17) can be proven exactly the same way using expression (37). For case  $m = 2$  ( $1/3 < X < 1/2$ ), we have

$$V = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(2X) \\ V_t(k-2X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(k-2X)}{2} \\ \frac{A(k-2X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Case  $m = 3$  ( $1/4 < X < 1/3$ ) gives us

$$V = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(2X) \\ V_t(3X) \\ V_t(k-3X) \\ V_t(k-2X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(k-3X)}{2} \\ \frac{A(k-3X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$



Finally, in case  $m = 4$  ( $1/5 < X < 1/4$ ), we have

$$V = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(2X) \\ V_t(3X) \\ V_t(4X) \\ V_t(k-4X) \\ V_t(k-3X) \\ V_t(k-2X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(k-4X)}{2} \\ \frac{A(k-4X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

For smaller  $X$  ( $m \geq 5$ ), we observe polynomials of higher degrees when calculating eigenvalues of matrix  $Q$ , and matrix decomposition turns out to be problematic. Nevertheless, the problem may be solved computationally for any given  $m$ .  $\square$

*Proof of Proposition 6.* Let's start with the possible deviation of storage. If the storage unit decides to enter the market and make a purchase under monopolistic quantities set by generators, it has to sell energy under Cournot quantities. The expected storage payoffs during one round-trip cycle are

$$-\left(\frac{1-a}{2} + k\right)k + \frac{\beta}{2-\beta} \frac{1+a-\delta k}{n+1} \delta k.$$

Storage doesn't gain any profits if

$$\frac{\beta}{2-\beta} \leq \frac{(n+1) \left(\frac{1-a}{2} + k\right)}{\delta(1+a-\delta k)},$$

which is equivalent to the right side of (18). Hence, as long as generators can maintain their collusive equilibrium, the storage unit should not purchase any energy and so never operates.

Let's analyze the possible deviations of a generator. First, we consider the case when

inequality (19) doesn't hold. It means that storage is not interested in entering the market with Cournot bids. A generator deviates from its monopolistic quantity when the shock is positive:  $\varepsilon = a$ . Let the deviation be  $\gamma$ . Then the generator's profit after deviating is

$$\left(\frac{1+a}{2n} + \gamma\right) \left(\frac{1+a}{2} - \gamma\right) + \frac{\beta}{1-\beta} \left(\frac{1}{2} \left(\frac{1+a}{n+1}\right)^2 + \frac{1}{2} \left(\frac{1-a}{n+1}\right)^2\right).$$

A generator increases its quantity by  $\gamma$ , which results in the price going down by  $\gamma$  also. Because of that, all other generators switch from monopolistic quantities to Cournot quantities, which results in the discounted expected payoff over infinite horizon expressed by the second item. This move is unprofitable if

$$\begin{aligned} \left(\frac{1+a}{2n} + \gamma\right) \left(\frac{1+a}{2} - \gamma\right) + \frac{\beta}{1-\beta} \left(\frac{1}{2} \left(\frac{1+a}{n+1}\right)^2 + \frac{1}{2} \left(\frac{1-a}{n+1}\right)^2\right) &\leq \\ &\leq \frac{1+a}{2n} \frac{1+a}{2} + \frac{\beta}{1-\beta} \left(\frac{1}{2} \frac{(1+a)^2}{4n} + \frac{1}{2} \frac{(1-a)^2}{4n}\right), \end{aligned}$$

which can be simplified to

$$\frac{n-1}{n} (1+a)\gamma - 2\gamma^2 \leq \frac{\beta}{1-\beta} \frac{(1+a^2)(n-1)^2}{2n(n+1)^2}.$$

The maximum on the left side can be achieved when  $\gamma_{max} = (1+a)(n-1)/(4n)$ . Then we obtain

$$\frac{(1+a)^2}{4n} \leq \frac{\beta}{1-\beta} \frac{(1+a^2)}{(n+1)^2}, \quad (38)$$

which is equivalent to the left side of (18).

Now assume that a generator deviates from its monopolistic quantity when the shock is negative:  $\varepsilon = -a$ . This move is unprofitable if

$$\begin{aligned} \left(\frac{1-a}{2n} + \gamma\right) \left(\frac{1-a}{2} - \gamma\right) + \frac{\beta}{1-\beta} \left(\frac{1}{2} \left(\frac{1+a}{n+1}\right)^2 + \frac{1}{2} \left(\frac{1-a}{n+1}\right)^2\right) &\leq \\ &\leq \frac{1-a}{2n} \frac{1-a}{2} + \frac{\beta}{1-\beta} \left(\frac{1}{2} \frac{(1+a)^2}{4n} + \frac{1}{2} \frac{(1-a)^2}{4n}\right), \end{aligned}$$

which can be simplified to

$$\frac{(1-a)^2}{4n} \leq \frac{\beta}{1-\beta} \frac{1+a^2}{(n+1)^2}.$$

This inequality is weaker than (38).

Now consider the case when inequality (19) holds. It means that storage finds profitable to enter the market immediately after any of the generators has deviated. Since the payoffs of a deviating generator with the participating storage unit are lower than the ones without it, the inequality (38) is sufficient to make this deviation unprofitable.

Finally, we should also prove that our pool of equilibrium strategies forms an SPNE even at information sets that are off the equilibrium path. Namely, generators must not want to deviate even if storage enters the market. Indeed, if the shock is positive, all the generators set static Cournot quantities, and it becomes unprofitable to deviate. If the shock is negative, any deviation from monopolistic quantities implies punishment that was already considered earlier. Hence, all the possible deviations of a generator are unprofitable, and the proposed strategies of all the players form Nash equilibrium.

To calculate the expected payoff of a generator, we obtain a recursive equation for generator's payoff  $Z_t$  starting from period  $t$ :

$$Z_t = \frac{1}{2}G_{01} + \frac{1}{2}G_{10} + \beta Z_{t+1}.$$

We can easily find  $Z_0 = U_g^0$  from this equation.

Since positive and negative shocks are equally likely, the cumulative consumers' expected payments per period are

$$C^0 = \frac{1}{2} \cdot \left( \frac{1+a}{2} \right)^2 + \frac{1}{2} \cdot \left( \frac{1-a}{2} \right)^2 = \frac{1+a^2}{4}.$$

□

*Proof of Proposition 7.* To justify inequalities (21), (22), and (23), we need to find the expected payoffs of the storage unit. Let the value function  $V_t^{\{-,+\}}(i)$ ,  $i \in \{0, k/2, k\}$  be the

total expected payoff of the storage operator from  $t$  on if the current state is empty ( $i = 0$ ), half-full ( $i = k/2$ ), or full ( $i = k$ ) and the current shock is either negative ( $-$ ) or positive ( $+$ ).

We have a system of recursive equations:

$$\begin{cases} V_t^-(k) = y \cdot \beta V_{t+1}^-(k) + (1-y) \cdot \left( A\left(\frac{k}{2}\right) + \beta V_{t+1}^+\left(\frac{k}{2}\right) \right), \\ V_t^-\left(\frac{k}{2}\right) = y \cdot \left( -B\left(\frac{k}{2}\right) + \beta V_{t+1}^-(k) \right) + (1-y) \cdot \left( A\left(\frac{k}{2}\right) + \beta V_{t+1}^+(0) \right), \\ V_t^+\left(\frac{k}{2}\right) = (1-x) \cdot \left( -B\left(\frac{k}{2}\right) + \beta V_{t+1}^-(k) \right) + x \cdot \left( A\left(\frac{k}{2}\right) + \beta V_{t+1}^+(0) \right), \\ V_t^+(0) = (1-x) \cdot \left( -B + \beta V_{t+1}^-\left(\frac{k}{2}\right) \right) + x \cdot \beta V_{t+1}^+(0). \end{cases}$$

for any integer  $t \geq 0$ . It can be rewritten in a matrix form

$$V_t = P_2 + \beta \cdot Q_2 \cdot V_{t+1}, \quad (39)$$

where

$$V_t = \begin{pmatrix} V_t^-(k) \\ V_t^-(k/2) \\ V_t^+(k/2) \\ V_t^+(0) \end{pmatrix}, \quad P_2 = \begin{pmatrix} (1-y)A(k/2) \\ (1-y)A(k/2) - yB(k/2) \\ xA(k/2) - (1-x)B(k/2) \\ -(1-x)B(k/2) \end{pmatrix}, \quad Q_2 = \begin{pmatrix} y & 0 & 1-y & 0 \\ y & 0 & 0 & 1-y \\ 1-x & 0 & 0 & x \\ 0 & 1-x & 0 & x \end{pmatrix}.$$

To calculate power  $t$  of matrix  $Q_2$ , we find the Jordan decomposition  $Q_2 = T \cdot J \cdot T^{-1}$  of

$Q_2$ . Here,

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{(1-x)(1-y)} & 0 & 0 \\ 0 & 0 & -\sqrt{(1-x)(1-y)} & 0 \\ 0 & 0 & 0 & x+y-1 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & -\frac{x\sqrt{1-y}}{y\sqrt{1-x}} & \frac{x\sqrt{1-y}}{y\sqrt{1-x}} & -\frac{1-y}{1-x} \\ 1 & \frac{\sqrt{(1-x)(1-y)}-x}{1-x} & -\frac{\sqrt{(1-x)(1-y)}+x}{1-x} & -\frac{1-y}{1-x} \\ 1 & \frac{x}{\sqrt{(1-x)(1-y)}} - \frac{x}{y} & -\frac{x}{\sqrt{(1-x)(1-y)}} - \frac{x}{y} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

so  $Q_2^t = T \cdot J^t \cdot T^{-1}$ .

For  $\beta < 1$ , we can find from (39) that

$$\begin{aligned} V_0 &= \sum_{i=0}^t \beta^i Q_2^i \cdot P_2 + \beta^{t+1} Q_2^{t+1} \cdot V_{t+1} \xrightarrow[t \rightarrow \infty]{} \sum_{i=0}^{\infty} \beta^i Q_2^i \cdot P_2 = \\ &= T \cdot \begin{pmatrix} \frac{1}{1-\beta} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\beta\sqrt{(1-x)(1-y)}} & 0 & 0 \\ 0 & 0 & \frac{1}{1+\beta\sqrt{(1-x)(1-y)}} & 0 \\ 0 & 0 & 0 & \frac{1}{1+\beta(1-x-y)} \end{pmatrix} \cdot T^{-1} \cdot P_2 = \\ &= \frac{1}{(1-\beta)(1-\beta d)(1-\beta^2(1-x)(1-y))} \times \\ &\times \begin{pmatrix} (1-y) \left( -\beta(1-x)(1+\beta^2 d) B\left(\frac{k}{2}\right) + (1-(1-y+xd)\beta^2) A\left(\frac{k}{2}\right) \right) \\ -((1-\beta y)(y-\beta d) + \beta^3(1-x)(1-y)d) B\left(\frac{k}{2}\right) + (1-y)(1-\beta x)(1+\beta^2 d) A\left(\frac{k}{2}\right) \\ -(1-x)(1-\beta y)(1+\beta^2 d) B\left(\frac{k}{2}\right) + ((1-\beta x)(x-\beta d) + \beta^3(1-x)(1-y)d) A\left(\frac{k}{2}\right) \\ (1-x) \left( -(1-(1-x+yd)\beta^2) B\left(\frac{k}{2}\right) + \beta(1-y)(1+\beta^2 d) A\left(\frac{k}{2}\right) \right) \end{pmatrix}. \end{aligned}$$

from where we finally get  $U_s^2 = V_0^+(0)$ :

$$U_s^2 = \frac{1-x}{(1-\beta)(1-\beta d)} \left( -B\left(\frac{k}{2}\right) + \beta(1-y)A\left(\frac{k}{2}\right) + \beta^2 y \frac{yB\left(\frac{k}{2}\right) + \beta x(1-y)A\left(\frac{k}{2}\right)}{1-\beta^2(1-x)(1-y)} \right).$$

Storage operates in this market only if  $U_s^2 > 0$ , which is exactly inequality (22).

For Condition 18, equation (39) reads

$$V_t = P_3 + \beta \cdot Q_3 \cdot V_{t+1},$$

where

$$V_t = \begin{pmatrix} V_t^-(k) \\ V_t^-(2k/3) \\ V_t^+(2k/3) \\ V_t^-(k/3) \\ V_t^+(k/3) \\ V_t^+(0) \end{pmatrix}, \quad P_3 = \begin{pmatrix} (1-y)A(k/2) \\ (1-y)A(k/2) - yB(k/2) \\ (1-y)A(k/2) - yB(k/2) \\ xA(k/2) - (1-x)B(k/2) \\ xA(k/2) - (1-x)B(k/2) \\ -(1-x)B(k/2) \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} y & 0 & 0 & 1-y & 0 & 0 \\ y & 0 & 0 & 0 & 1-y & 0 \\ 0 & y & 0 & 0 & 0 & 1-y \\ 1-x & 0 & 0 & 0 & x & 0 \\ 0 & 1-x & 0 & 0 & 0 & x \\ 0 & 0 & 1-x & 0 & 0 & x \end{pmatrix}.$$

The six eigenvalues of matrix  $Q_3$  that compose the diagonal of the corresponding Jordan matrix  $J$  are

$$\lambda_1 = 1, \quad \lambda_2 = x + y - 1, \quad \lambda_{3,4,5,6} = \pm \sqrt{(1-x)(1-y) \pm \sqrt{xy(1-x)(1-y)}}.$$

Following the same argumentation as in case of  $k/2$ , we get

$$V_0^+(0) = \frac{1-x}{(1-\beta)(1-\beta d)} \left[ -B\left(\frac{k}{3}\right) + \beta(1-y)A\left(\frac{k}{3}\right) + \beta^3 y \frac{y^2 B\left(\frac{k}{3}\right) + x(1-y)(1+\beta^2 d)A\left(\frac{k}{3}\right)}{(1-\beta^2(1-x)(1-y))^2 - \beta^4 xy(1-x)(1-y)} \right].$$

Inequality  $V_0^+(0) = U_s^3 > 0$  is exactly (23).

Four possible deviations of the storage unit should be considered. All other deviations are just compositions of those four.

- A storage unit that is not empty deviates by not selling under the positive shock. Then there may be only loss comparing to the default strategy. Indeed, nothing changes on the market except the future profits to be discounted by  $\beta$ .
- A storage unit that is not full deviates by not buying under the negative shock. Also, no gains here.
- A full storage unit deviates by selling under the negative shock. In this situation, the quantities supplied by the generators are  $q = (1-a)/(n+1)$ . The resulting price after the deviation are

$$p = 1 - a - \delta \frac{k}{m} - n \frac{1-a}{n+1} = \frac{1-a}{n+1} - \delta \frac{k}{m}.$$

To make this deviation profitable, the storage operator must gain more than if it waits for the positive shock and sells in that period, but this contradicts (22) or (23).

- The nonfull storage unit deviates by buying under the positive shock. Here we have  $q = (1+a)/(n+1)$ , and the resulting price after the deviation is  $p = (1+a)/(n+1) + k/m$ . It is easy to verify that the corresponding payoff is strictly negative.

Next we must rule out possible deviations of the generators. In each round, we have a static Cournot equilibrium for all the participants. Any change of the equilibrium quantity in round  $t$  leads to decreasing the payoffs in that round and, thus, decreasing the overall payoffs.  $\square$

### A.3 Optimal Bid Values

For each bunch of parameter values, we can find the optimal proportion  $r_{opt}$  and, hence, the optimal initial bid  $r_{opt}k$  for the case of proportional bids. Also, we can find the optimal constant bid  $X$ . We stay with the parameter values used in our examples in the main text with  $n = 2$ ,  $\beta = \delta = 0.95$ . Table 4 presents the results.

	Proportional bids, $a = 0.6$					Proportional bids, $a = 0.2$				
$k$	0.25	0.45	0.65	0.85	1.05	0.1	0.15	0.2	0.25	0.3
$r_{opt}$	0.91	0.68	0.543	0.451	0.385	0.795	0.635	0.526	0.447	0.385
$r_{opt}k$	<b>0.228</b>	<b>0.306</b>	<b>0.353</b>	<b>0.384</b>	<b>0.404</b>	<b>0.08</b>	<b>0.095</b>	<b>0.105</b>	<b>0.112</b>	<b>0.115</b>
	Constant bids, $a = 0.6$					Constant bids, $a = 0.2$				
$k$	0.25	0.45	0.65	0.85	1.05	0.1	0.15	0.2	0.25	0.3
$X$	<b>0.21</b>	<b>0.259</b>	<b>0.253</b>	<b>0.248</b>	<b>0.299</b>	<b>0.068</b>	<b>0.066</b>	<b>0.074</b>	<b>0.069</b>	—

Table 4

Some intuitive conclusions we can derive from this table:

- Bids decrease along with the shock in both cases;
- The first proportional bids are always higher than constant bids. This owes to more flexibility towards the boundaries – the asymptotic behavior of the linear heuristic.
- The initial proportional bids tend to increase with  $k$ , but slowly. It is not the case for constant bids because the continuation risk is more salient under this heuristic, and so kicks in sooner. Indeed, constant bids increase as long as a number of steps (parameter  $m$  from chapter 4.1.4) is the same and decrease with the next shift.



# References

- Acemoglu, Daron, Ali Kahkbad, and Asuman Ozdaglar (2017) “Competition in Electricity Markets with Renewable Energy Sources,” *The Energy Journal*, 38, 137–155.
- Andres-Cerezo, A. and Natalia Fabra (2023) “Storing power: market structure matters,” *Rand Journal of Economics*, 54 (1), 3–53.
- Bonatti, A., G. Cisternas, and Juuso Toikka (2017) “Dynamic Oligopoly with Incomplete Information,” *The Review of Economic Studies*, 84, 503–546.
- Borenstein, Severin and James Bushnell (1999) “An empirical analysis of the potential for market power in California’s electricity industry,” *Journal of Industrial Economics*, 47 (3), 285–323.
- Borenstein, Severin, James Bushnell, and C. R. Knittel (1999) “Market Power in Electricity Markets: Beyond Concentration Measures,” *Energy Journal*, 20 (4).
- Bushnell, James, Erin Mansur, and Celeste Saravia (2008) “Vertical Arrangements, Market Structure and Competition: An analysis of Restructured U.S. Electricity Markets,” *The American Economic Review*, 98 (1).
- Butters, R. Andrew, Jackson Dorsey, and Gautam Gowrisankaran (Working Paper) “Soaking Up the Sun: Battery Investment, Renewable Energy, and Market Equilibrium,” *R&R Econometrica*.
- Chatterjee, K., Martin Jurdzinski, and Rupak Majumdar (2003) “On Nash equilibria in stochastic games,” *Defense Technical Information Centre*.
- Geske, J. and Richard Green (2020) “Optimal Storage, Investment and Management under Uncertainty: It is Costly to Avoid Outages!,” *Energy Journal*, 41 (SI).
- Glebkin, Sergei, Semyon Malamud, and Alberto Teguia (Forthcoming) “Illiquidity and higher cumulants,” *The Review of Financial Studies*.

- Graham, Ronald L., Donald E. Knuth, and Patashnik Oren (1994) *Concrete Mathematics: a foundation for computer science*: ADDISON-WESLEY PUBLISHING COMPANY.
- Harrison, J. M. and Alison Taylor (1978) “Optimal control of a Brownian storage system,” *Stochastic processes and their applications*, 6, 179–194.
- Karaduman, Omer (2020) “The economics of grid-scale storage,” *mimeo*, Stanford University.
- Klemperer, P. and Margaret Meyer (1989) “Supply function equilibria in oligopoly under uncertainty,” *Econometrica*, 57 (6), 1243–1277.
- Lundin, Erik and Thomas P. Tangerås (2020) “Cournot competition in wholesale electricity markets: The Nordic power exchange, Nord Pool,” *International Journal of Industrial Organization*, 68.
- Schmalensee, Richard (2022) “Competitive Energy Storage and the Duck Curve,” *Energy Journal*, 43 (2).
- Shapley, Lloyd (1953) “Stochastic games,” *Proceedings of the National Academy of Sciences*, 39, 1095–1100.
- Vayanos, Dimitri (1999) “Strategic trading and welfare in a dynamic market,” *The Review of Economic Studies*, 66, 219–254.
- Willems, Bert, Ina Rumiantseva, and Hannes Weigt (2009) “Cournot versus Supply Functions: What does the data tell us?” *Energy Economics*, 31 (1), 38–47.
- Williams, Olayinka and Richard Green (2022) “Electricity storage and market power,” *Energy Policy*, 164.