Near-optimal Storage Strategies in Electricity Markets

Sergei Balakin* and Guillaume Roger*

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Abstract

Storing electricity is completely essential to the energy transition. It also deeply disrupts the manner in which electricity markets operate, for it introduces delay. In this paper, we consider a dynamic model of an oligopolistic market with demand shocks, in which a storage unit buys and sells energy subject to a capacity constraint. To make progress in this stochastic game, we restrict attention to simple heuristics, and we can characterise the optimal policy of a storage unit in this restricted class of heuristics. The heuristics, the exogenous stochastic process and the capacity constraint interact to induce rich dynamics. The optimal policy is sensitive to the nature of demand shocks and to storage capacity. Capacity utilisation decreases with capacity size for two reasons. First, the storage unit internalises its unilateral market power, yet has to remain large enough to not render the arbitrage trade trivial. Second, uncertainty is costly: a storage operator does not want to be stuck empty or full, and so limits the quantities traded in any interval. Too large a capacity nullifies the arbitrage spread, even with the optimal trading policy. We construct an equilibrium, in which electricity arbitrage is never profitable, and so conclude that successful entry is not a foregone conclusion.

This work informs market participants as well as the design of electricity markets with storage. It is particularly relevant to major markets with rapid penetration of renewable energy sources, like California or Australia. It can also be applied to trading securities.

Key words: *stochastic game, dynamic trading, energy storage*

JEL: *C73, D43, D47, Q41, Q42*

*Monash University. Corresponding author: Roger – guillaume.roger72@gmail.com

1 Introduction

Producing electricity on a large scale and securely so as to transition away from polluting fossil fuels requires vast amounts of storage. That storage allows energy to be available when it is needed, say at night, rather than when it is produced, say on a sunny afternoon. Hence storage delivers the intertemporal smoothing of consumption and production that is completely essential to effecting the energy transition. However we know very little of the economics of electricity storage because for long it was simply not an option. This paper addresses this gap – in part.

We study a model of electricity trading based on storage over a long horizon, and rooted in an oligopolistic market. Conventional generators with identical technology produce quantities for immediate sale and demand is subject to idiosyncratic and symmetric shocks (while mean demand is constant).[1] A storage operator can step in and implement the simple idea of "buying low and selling high", the details of which are in quite complicated and rich. In this stochastic game (Shapley (1953)), which admits a very large number of equilibria, we must limit ourselves to studying simple heuristics rather than the more desirable equilibrium strategies that remain out of reach. However, with this restriction, we can characterise optimal heuristics in their class, and study some comparative statics. As in any stochastic game, the strategies interact with the exogenous stochastic process to induce endogenous transitions between states. Here, this endogenous stochastic process is further enriched by constraints on capacity and on the initial condition to generate novel dynamics.

From our simple model we learn that a storage operator must balance three competing forces: its unilateral market power (quantity), managing the arbitrage spread on which it feeds (capacity), and continuing (the value of future trades). On the first account, a storage unit with large enough a capacity internalises its own market power and thus withholds quantities. On the second one, it must have enough capacity to fully exploit the intertemporal energy arbitrage, but not too much so as to not nullify the arbitrage spread – since it has market power. Finally, it must always trade quantities that remain large enough so that the continuation value

[1] One can also add renewable energy with stochastic supply and conceive of the demand function as residual demand without material consequences.

does not vanish *in finite time*, yet not so large as to nullify the arbitrage spread. We model Cournot competition rather than relying on the more elegant, but completely intractable, supply-function equilibrium.[2] We also make no distinction between power and energy; it is as if a quantity were either energy or power for a prescribed duration (e.g. for the trading interval). Even then, we must first focus on a behaviour of the conventional generators and a heuristic of the storage operator to make progress. For two simple heuristics, we are able to uncover a recursive structure that is tractable and allows us to compute the corresponding value function, which can then be optimised. The simple independent case can be extended to a richer, Markovian shock structure with more or less persistence in shocks, however only for a restricted subset of these heuristics.

The success of a nascent storage industry cannot be taken for granted. Indeed we are also able to construct a collusive equilibrium, in which generators are able to act tacitly to prevent a storage unit from operating. More precisely, they are able to render arbitrage unprofitable if the storage operator ever decides to incur the charging cost. While this can be overcome with government intervention – for example, a simple subsidy for the first charge in this model, it shows that some measure of support may be necessary to a successful entry.

Storage has been in existence for some time in the form of hydro-electric power. However storing water to generate electricity differs from having to first *purchase* electricity in order to sell it later. Once a dam is built, the water inflow is free, exogenous and stochastic; in contrast, a storage unit pays for the energy it buys, it can have (a measure of) monopsony power, and it makes that decision optimally as part of its trading strategy. Furthermore, most models of dam management amount to an optimal control problem rather than a game, and ignore completely the market power of the dam operator on electricity prices. We show that both market power and having to buy energy are first-order considerations for a storage operator.

The model we present in this paper can be used, possibly with some adaptations, to trade securities; indeed, this intermediation activity shares many characteristics with trading electricity through storage: assets are bought and sold, a revenue is generated by a (mostly

[2] Recall that in an SFE the Cournot outcome constitutes an upper bound for the payoffs to suppliers Klemperer and Meyer (1989).

intertemporal) spread, holding inventory (storing) and the optimal timing of trades are important, and price impact matters a great deal. As far as we can tell, no such model of securities trading has yet been written.[3] This work is also conceptually connected to the inventory management problem; see Harrison and Taylor (1978) for example. However that problem is strictly one of stochastic control – not a game, in which the per-unit payoffs (rewards or costs) are strictly exogenous. In our model, prices are determined endogenously of course, which gives rise to the trade-offs we mentioned earlier.

This paper is one of very few on the economics of electricity storage. Karaduman (2020) is the first to study grid scale storage, using Australian data from the National Electricity Market and focusing on the activities of the Hornsdale Power Reserve (HPR). Generators and the storage unit play an infinite horizon game and market power is internalised. However, Karaduman (2020) does not compute the best reply; rather he simulates it from the data. Hence the actual behaviour of the storage unit is never known. Karaduman shows that the HPR cannot profitably exploit arbitrage opportunities.[4] Andres-Cerezo and Fabra (2022) study the question of market structure with storage, but leave aside how storage actually behaves. They ask in particular "who should (or not) own storage units", and find that market structure matters a great deal. Specifically, a generator should not own a storage unit as it enhances market power, especially at time when demand is the highest. That is, in those times the substitutability between storage and generation should be exploited to its fullest, but the joint ownership of these two assets induces more quantity withholding thanks to (enhanced) market power. Schmalensee (2022) studies storage investment in a competitive model of an electricity market. He models the idea of intertemporal smoothing of energy availability, that is, the intra-day behaviour of storage rather than short-term arbitrage opportunities. Energy generation and storage are competitive rather than oligoplistic, and he focuses on capacity investment, which we take as exogenous. He also assumes that storage is fully discharged after the "nighttime", while we let the storage operator make that decision in equilibrium. Ekholm and Virasjoki (2020) present model of imperfect competition with 100% renewable energy

[3]To be sure, there are models of securities trading; take Rosu (2009) for example. However the dynamics are essentially trivial in that agents leave the market after trading; not so here.

[4]HPR generates other revenues, which more than cover its costs; but it fails on energy arbitrage.

generation and storage owned by renewable generators, however without uncertainty. There too, storage is not really used for arbitrage but for intertemporal smoothing. The work closest to ours is that of Geske and Green (2020), who do study arbitrage in a model of imperfect competition with demand uncertainty and diurnal, weekly and seasonal patterns. In such a complicated environment they must limit themselves to numerical (approximate) solutions to the welfare maximization problem. They also exhibit precautionary behaviour – in our case, quantity withholding. Our stochastic environment is simpler and allows us to make some progress analytically. Finally, in a very different strand of literature, Bonatti et al. (2017) study a dynamic Cournot model under incomplete information, in which agents learn about the cost of its competitors through play. The equilibrium converges to the repeated static Nash equilibrium. We take this as a justification for focusing on this equilibrium at the outset.

2 Model

Consider a very simple dynamic market with one storage unit, n electricity generators labelled $j = 1, 2, \ldots n$, and a pool of consumers. We simplify institutional details so that retailers and consumers are confounded and retailing has no cost; another way of saying this is that retailers perfectly reflect the behavior of consumers. The behavior of consumers is described by the demand function $D(p_t, \varepsilon_t)$ for each period t, where ε_t is a shock distributed according to some commonly known distribution F. Each of the generators j produces a quantity of energy q_t^j for each period t, and may or may not be subject to capacity constraints. The storage unit has finite capacity k. In each period, it can either buy energy (charge) up to its capacity, or sell any amount of available energy (discharge). This process can be described formally by a simple equation of motion:

$$c_t = c_{t-1} + b_t - \frac{s_t}{\eta}, \qquad t \in \mathbb{N}, \qquad c_0 = 0. \tag{1}$$

Here, c_t is a current level of charge ($0 \leq c_t \leq k$), η is a round-trip efficiency parameter ($0 < \eta \leq 1$), and $b_t \geq 0, s_t \geq 0$. A storage operator can only either buy or sell in each period, so $b_t \cdot s_t = 0$ for any t – this is a technical characteristic. In each period, the market clears if

$$D(p_t; \varepsilon_t) = \sum_{j=1}^{n} q_t^j - b_t + s_t$$

for any t, where we suppose that players engage in Cournot competition. Since the nature of competition is not the primary object of interest, throughout the rest of the paper we consider a linear demand function:

$$D(p_t; \varepsilon_t) = 1 - p_t + \varepsilon_t.$$

Rather, the goal is to find optimal policies $\{b_t; s_t\}_{t=0}^{\infty}$ which are the part of a dynamic Nash equilibrium. We suppose the storage unit has a discount factor $\beta < 1$; it is exposed to a strictly positive interest rate. We note that depending on the decisions of the storage operator, in each round there may be either

(a) n (symmetric) competitors or

(b) $n + 1$ competitors with a limited capacity for one of them.

First we characterise the optimal variables of a static problem, which are useful throughout.[5]

Lemma 1. *If the storage unit is a seller with c units of energy available, then the (symmetric) equilibrium price p^* and equilibrium quantities s^* and q^* under Cournot competition are:-*

$$p^* = \frac{1 + \varepsilon - c}{n + 1}; \qquad s^* = c; \qquad q^* = \frac{1 + \varepsilon - c}{n + 1} \qquad \text{if } c \leq \frac{1 + \varepsilon}{n + 2}; \tag{2}$$

$$p^* = \frac{1 + \varepsilon}{n + 2}; \qquad s^* = q^* = \frac{1 + \varepsilon}{n + 2} \qquad \text{if } c > \frac{1 + \varepsilon}{n + 2}. \tag{3}$$

Proof. In the first case, we have Cournot competition between $n + 1$ players where one of the players has limited capacity. Standard Cournot competition between $n + 1$ players is observed in the second case.

[5] We omit the subindex t if it does not lead to confusion.

Lemma 2. *If the storage unit is a buyer with willingness to purchase c units, then the (symmetric) equilibrium price p^* and equilibrium quantities b^* and q^* under Cournot competition are*

$$p^* = \frac{1+\varepsilon+c}{n+1}; \qquad b^* = c; \qquad q^* = \frac{1+\varepsilon+c}{n+1}. \tag{4}$$

Proof. Here we have standard Cournot competition between n players with increased demand.

If the storage unit neither buys nor sells, standard Cournot competition between n generators prevails, and then $p^* = q^* = (1+\varepsilon)/(n+1)$ in the symmetric equilibrium. Next we turn to the object of this paper, which is trading in the dynamic game. We begin with a special case as an example.

3 An introductory example

Let the shocks ε_t be independently and identically distributed,

$$Pr\{\varepsilon = a\} = Pr\{\varepsilon = -a\} = 1/2; \qquad 0 < a < 1 \tag{5}$$

for any t. Suppose also that the storage operator can only either charge or discharge in full; it can buy either 0 or k units of energy, or sell either 0 or $-k$ units at each period of time. (Thus, option (3) is not available.) For convenience, we define charging costs (when purchasing energy) under the negative shock as B and likewise the revenue it earns when selling energy under the positive shock as A:

$$B = B(k) = \frac{1-a+k}{n+1}k; \qquad A = A(k) = \frac{1+a-k}{n+1}k.$$

Observe that it cannot be optimal to charge when the shock $''$ is positive, nor can it be optimal to discharge when it is negative. Let also the coefficients

$$G_{01} = \left(\frac{1+a}{n+1}\right)^2; \quad G_{00} = \left(\frac{1 \quad a+k}{n+1}\right)^2; \quad G_{10} = \left(\frac{1 \quad a}{n+1}\right)^2; \quad G_{11} = \left(\frac{1+a \quad k}{n+1}\right)^2:$$

Here, G_{ij} is a (non-discounted) generator's payoff when the storage is either empty ($i = 0$) or full ($i = 1$) and when the shock is either negative ($j = 0$) or positive ($j = 1$). Suppose also the discount factor is such that

$$B < \frac{\quad}{2} A:$$

Then a dynamic equilibrium exists and is characterised as follows:[6]

- the empty storage buys k units with the first negative shock and sells k units with the first positive shock afterwards;
- in each period, the generators set quantities q according to static Cournot competition and based on the current shock and the state of the storage (full or empty). Namely,

$$\text{when storage is empty,} \quad q \;= \frac{1+a}{n+1} \text{ if } '' = a; \quad q \;= \frac{1 \quad a+k}{n+1} \text{ if } '' = \quad a;$$
$$\text{when storage is full,} \quad q \;= \frac{1 \quad a}{n+1} \text{ if } '' = \quad a; \quad q \;= \frac{1+a \quad k}{n+1} \text{ if } '' = a:$$

Cumulative consumers' expected payments per period C_1 are

$$C_1 = \frac{1}{2(n+1)^2}\left(2n(1+a^2) \quad ka(n \quad 1) \quad k^2\right):$$

Expected payoff of the storage U_s and the generators U_g takes the following form:

$$U_s = \frac{1}{2}\left[\quad B + \frac{\quad}{2(1 \quad)}(A \quad B)\right];$$
$$U_g = \frac{1}{2}(G_{01} + G_{00}) + \frac{\quad}{4(1 \quad)}(G_{10} + G_{11} + G_{00} + G_{01}):$$

[6] We prove this claim formally in the Appendix.

Of course, this equilibrium is not the only one, but it is one that can be used to explore some of, but not all, the salient features of the general problem. First, we see that storage increases the output of generators when the demand shock is negative; this also increases prices (when they are otherwise low). Then storage is a ***complement*** to generators. Storage also decreases the output of these generators when the shock is positive; this concurrently depresses otherwise high prices. Now storage is a ***substitute*** for generators. These patterns are more than astute observations if considering entry of new generation capacity; for solar generation, for example, storage is a strict complement but for thermal generators, it is both. This may matter for decisions such as entry, or for questions of competition policy.

Second, a storage unit starts empty and must always first buy energy. This is apparent in its profit function U_S. The total payoff in this simple example, is the discounted present value of the spread on energy sales ($A \quad B$) net of the first charge $\quad B$. In a richer environment, this profit is "eroded" by the cost of uncertainty and the impact of unilateral market power.

Finally, this example is the closest we can get to a benchmark: indeed, intertemporal arbitrage only makes sense in a dynamic model; there is no static equivalent.

4 Trading energy over the long horizon

As we know from the literature on repeated games and on stochastic games (see, for example, Chatterjee et al. (2003)), the game described in Section 2 admits a large number of equilibria. Short of constructing equilibria that exhibit features the analyst seeks, it is impossible to characterise the optimal strategy of a storage unit. To overcome this problem, we focus instead on simple heuristics, which allows us to ***compute*** the value function of the storage operator for that heuristic. Then we can find the optimal level of this simple heuristic, and engage in comparative statics. In the example of Section 3, the heuristic is trivial: charge and discharge in full at any opportunity. In what follows, we explore richer heuristics, where capacity need not be used in full at every opportunity.

Throughout we continue focusing on the repetition of the Cournot equilibrium stage game – see Lemmata 1 and 2. As we know, this equilibrium delivers the lowest payoffs to sellers, and entails the least quantity distortions and the lowest prices. Hence our analysis can be taken as a lower bound on what can be achieved by a storage operator.

4.1 Independent binary shocks

We stay with the simple independent shock structure $(1/2; 1/2)$ to begin with, which affords us some tractability, even though how much to discharge is now endogenous. The objective of the storage operator is

$$\max_{s;b} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t p_t(s_t - b_t)\right]; \tag{6}$$

subject to the law of motion (1) and the important capacity constraint

$$0 \leq c \leq k. \tag{7}$$

Let $V_t(c)$ be the total expected payoff of the storage from moment t if the current state is c.[7] Then the recursive equation may be written in the following form:

$$\begin{cases} V_t(c) &= \frac{1}{2}\left(-\frac{1-a+b(c)}{n+1}b(c)+\beta V_{t+1}(c+b(c))\right) \\ &+\frac{1}{2}\left(\frac{1+a-s(c)}{n+1}s(c)+\beta V_{t+1}(c-s(c))\right); \\ V_t(0) &= \frac{1}{2}\left(-\frac{1-a+b(0)}{n+1}b(0)+\beta V_{t+1}(b(0))\right); \\ V_t(k) &= \frac{1}{2}\left(\frac{1+a-s(k)}{n+1}s(k)+\beta V_{t+1}(k-s(k))\right); \end{cases} \tag{8}$$

where $0 \leq b(c) \leq k - c$ and $0 \leq s(c) \leq c$ for any c. The function $b(c)$ is how much the storage would like to buy if its state of charge is already c and the shock is negative; $s(c)$ is how much the storage with its state of charge c would like to sell when facing the positive shock. We aim to find functions $b(c)$ and $s(c)$ that maximize $V_0(0)$ subject to (1) and (7).

[7] We dispense proving that the Dynamic Programming Principle holds in this environment, which is quite standard.

4.1.1 Heuristic 1: proportional bids (constant fraction)

Our first heuristic calls for constant charging and discharging fractions. For example, starting from empty, if the chosen fraction is $1/2$, the storage operator charges $k/2 = c$, which is now the state of charge. If facing another negative shock, she charges $(k - c)/2 = k/4$, now the state of charge is $c = 3k/4$. If facing a positive shock instead, she discharges $c/2 = k/4$, so the state of charge is $c = k/4$. And so on. More formally, $b(c) = r(k - c)$ or $s(c) = rc$, respectively ($0 \le r \le 1$). This problem is rendered complicated because of the constraint $0 \le c \le k$, and the fact that the grid of the state space grows exponentially (if $r < 1$). Nonetheless there exists a recursive structure that can be exploited.

In addition, a storage unit cannot start from an arbitrary state, but it must commence at $c = 0$. Hence it can be stuck at that level for some period before being able to charge and start trading. Let $b(c) = r(k - c)$ and $s(c) = rc$ and

$$B(rk) = \frac{1 - a + rk}{n+1} - rk; \qquad A(rk) = \frac{1 + a - rk}{n+1} - rk.$$

Proposition 3. *The overall expected profit of the storage is*

$$\begin{aligned} U_s^P &= \frac{1}{2[1 - (1 - r)\ ]}\left(-B(rk) + \frac{r}{2(1 - \ )}\left(-B(rk) + A(rk)\right)\right) \\ &+ \frac{}{1 - }\,\frac{k^2 r^3 (1 - r)(1 + {}^2)}{4(n+1)\left(1 - (1 - r)^2\ \right)}. \end{aligned} \tag{9}$$

The proof of this Proposition, as all others, is relegated to the Appendix, Section A.2.

Expression (9) entails three elements, abstracting from the very first term that is a modified discount factor. The first term in the bracket is the cost of the initial charge. The second term is the discounted arbitrage profit from the first trade onward; this is the simple "buy low, sell high" mantra. The last term is strictly positive and not at all connected to arbitrage since it is independent of a, and r is only linearly connected to k (so its optimal is independent of k). It represents the benefit of flexibility in the face of uncertainty. Indeed, when $r = 1$, as in the introductory example of Section 3, this last term is zero, and all the weight is assigned to the simple arbitrage revenue. This term is largest for some interior value, which captures this notion of flexibility.

The payoff given in (9) is expressed in terms of the capacity k of the storage unit and its choice of heuristic r, as are the quantities $A(rk)$ and $B(rk)$. This allows us not only to find the optimal proportion r to maximise U_s^P, but also to engage in comparative statics with respects to k. The first-order condition of (9) does not lend itself to easy manipulation nor interpretation, but Condition (9) can be graphed. To this end, let $n = 2$, $\quad = 0{:}95$, and $\quad = 0{:}95$. Consider payoffs values for different capacities and shock magnitudes. Figure 1 corresponds to high shocks $a = 0{:}6$ with capacity k moving from 0.15 to 1.15, and Figure 2 corresponds to low shocks $a = 0{:}2$ where k changes from 0.05 to 0.3.

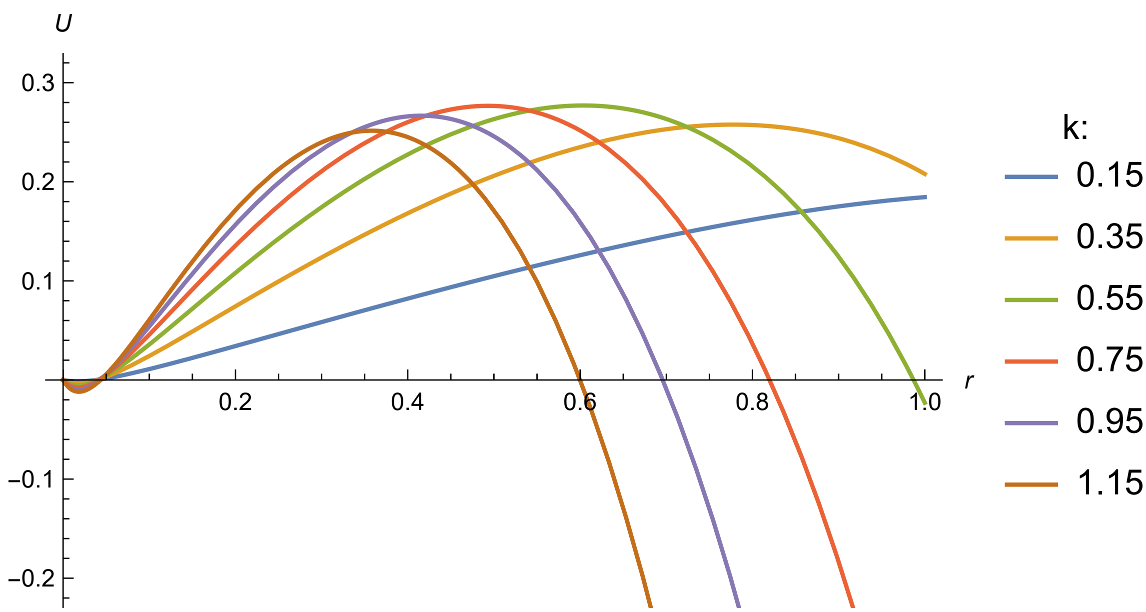

Figure 1: Payoff functions $U_s^P(r)$ for different capacities k when the magnitude of the shock is high enough: $a = 0.6$.

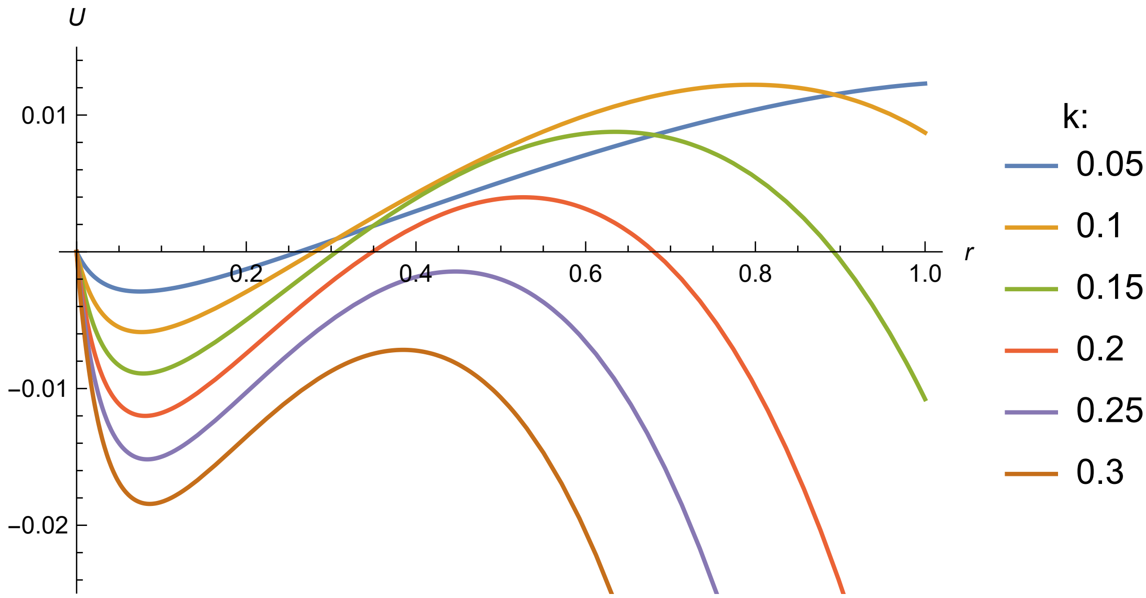

Figure 2: Payoff functions $U_s^P(r)$ for different capacities k when the magnitude of the shock is low: $a = 0.2$.

First we see that for relatively small capacities (k=a), it is optimal to charge and discharge in full at each opportunity. This is the heuristic of our introductory example. Second, from the lowest relative capacity, the maximum of the payoff function U_s^P increases as capacity expands, however only to a point. Third, concurrently, as relative capacity increases, the optimal proportion r decreases: large storage units use less of their capacity in any single trade.

There are subtleties to these two findings. For an arbitrary heuristic r, a large (relative) capacity nullifies the spread $A(rk) \quad B(rk)$: this is the price effect (or market power effect). In turn, for a fixed, large enough (relative) capacity, a storage operator internalises its own (unilateral) market power, and so she uses less capacity as that capacity increases; hence the function $r(k)$ decreases in k. This is also apparent from (9), where the arbitrage term $(A \quad B)$ rapidly decreases in k, while the last term ("flexibility") increases in k. Given k, the storage operator chooses to decrease r.

We also see that very small r deliver no profit at all even though there are no fixed costs; this is apparent from (9) as well: at r close to zero, there is neither arbitrage nor flexibility. These "frictions" can be explained too: when r is very small, the quantities traded keep decreasing rapidly and become negligible in finite time. This nullifies the arbitrage spread

$A(rk) \quad B(rk)$ and the continuation value *after* the first charge rapidly becomes negligible, but that first charge is a cost. Hence, both the heuristic r and the capacity choice k must remain interior.

Finally, very small shocks cannot sustain the operations of a storage unit, as we see from Figure 2. For a relatively large capacity, a storage unit can only make losses – and so should not enter the market. This is easy to understand: with small shocks ϵ, the spread can only be small. Then a (relatively) large capacity easily nullifies that spread for any choice of r – even though the relative capacity k/a is almost constant for each choice k across Figures 1 and 2. This can also be see from the profit function (9), which is linear in a but quadratic (negative) in k.

4.1.2 Heuristic 2: constant quantities

Here the storage operator buys or sells a constant quantity X (e.g 10MW) each period, starting from empty as well. To avoid having to deal with partial fills at the boundaries, we let $X := k/m$, so m is the number of steps to move from empty to full. In the set of admissible strategies, restricting m to be an integer may not be fully optimal, but we expect the corresponding loss to be small – if it exists.

The optimal behaviour differs from the proportional case. Under the proportional heuristic, boundaries are never reached: the storage unit can never be completely full nor completely empty in finite time. But here it becomes completely empty or completely full with positive probability. This induces rich dynamics that we label "waves". Handling these waves is the main challenge in this otherwise simple environment. To see this, first let

$$B\left(\frac{k}{m}\right) = \frac{1 \quad a + k/m}{n+1} \quad \frac{k}{m}; \qquad A\left(\frac{k}{m}\right) = \frac{1 + a \quad k/m}{n+1} \quad \frac{k}{m}.$$

Now consider a standard binomial tree representing the state space as drawn in Figure 3.

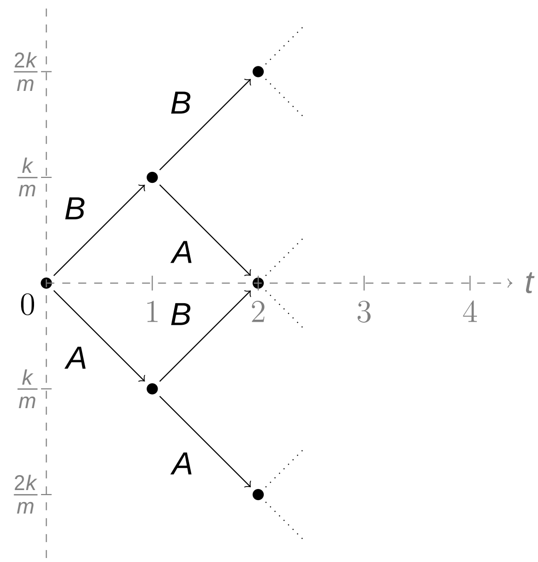

Figure 3: No boundaries. A and B available in any point.

Starting from zero, truncate this binomial tree from below – this leaves the top half of the tree, as in Figure 4. In this Figure, the light gray area is the region where the weights that cannot go down start going up instead. This changes the probabilities of reaching any node; for example, the point with coordinates $(1;0)$ can be reached from the preceding node $(0;0)$, whereas in unconstrained the binomial tree (Figure 3), it can ***never*** be reached. Likewise for the point $(2;1)$, which can be reached from $(1;0)$ in the truncated tree but never in the unconstrained tree. In turn this affects the probability of reaching $(3;1)$, which is accessible in both cases. The states with affected probabilities are marked with a thicker dot.

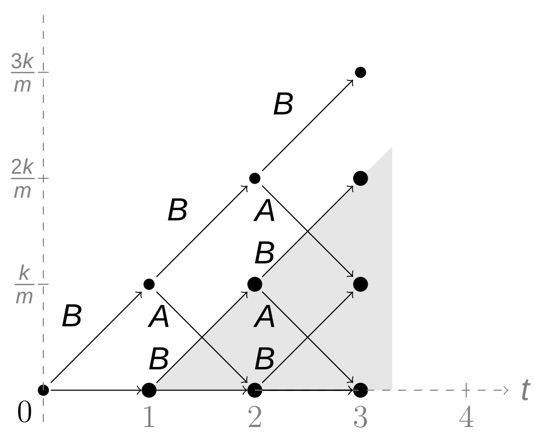

Figure 4: One bound – 0. There is no A in state $c = 0$.

Then truncate this tree further from above at the capacity level k to create a tunnel. This is depicted in Figure 5. The admissible state space is limited to that tunnel, in which we already know either new states can be reached, or some states can be reached with different probabilities. This upper boundary modifies the state space in much the same way, but in the other direction. Hitherto unreachable states can be reached, and the probability mass on already reachable states can be changed. In Figure 5 the darker gray area represents a second region, in which the upper bound k becomes active and forces the probability mass back down; hence the waves. This process continue on over the infinite horizon, and the successive reflections at the boundaries perpetuate these waves.

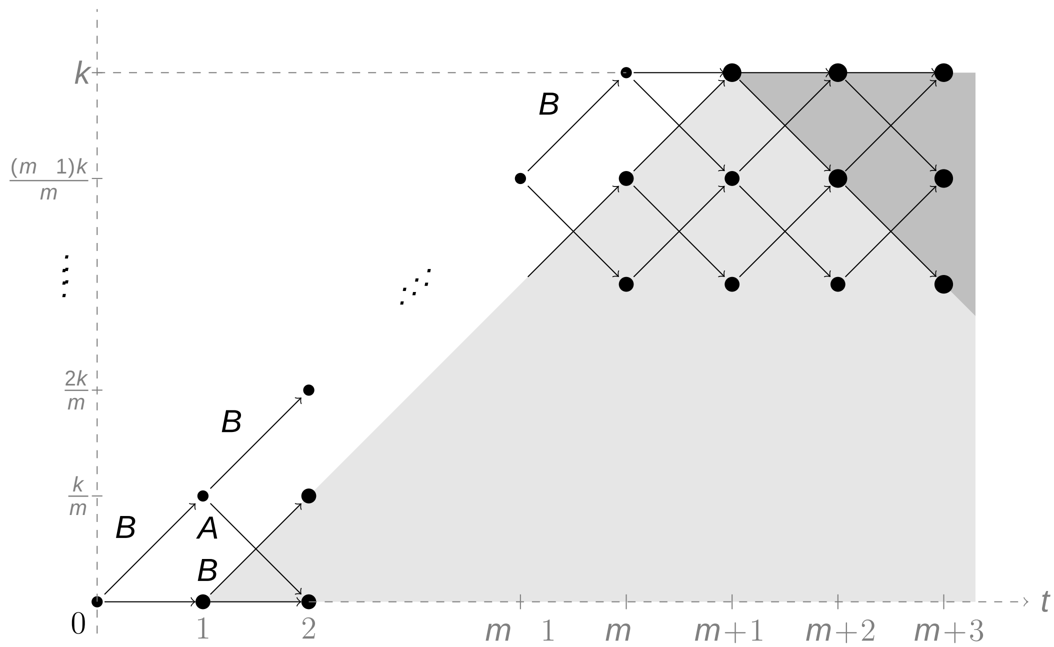

Figure 5: Two bounds – 0 and k. Here there are three types of thickness of points (states), depending on how many waves affect the corresponding probability (here, 0, 1, or 2). There is no A in state $c = 0$ and no B in state $c = k$.

These waves are periodic, which suggests a recursive structure can be uncovered. We are able to exploit this and compute the value function of the storage operator for this heurisitc too.

Proposition 4. *The overall expected pro t of the storage operator is*

$$
\begin{aligned}
U_s^C \;=\;& \frac{B(k{=}m) + \quad A(k{=}m)}{2(1 \qquad)} \quad \frac{A(k{=}m)}{} \sum_{i=1}^{1} \left(\frac{}{2}\right)^{2i} \left(C_{2i}^{i} \, {}^{1}{}_{1} + \frac{}{2} C_{2i}^{i} \right) \\
&+ \frac{1+}{} \left(B\left(\frac{k}{m}\right) \sum_{j=0}^{1} \left(\frac{}{2}\right)^{(m+1)(2j+1)} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{2i} C_{2i+(m+1)(2j+1)}^{i} \right. \\
&\qquad \left. A\left(\frac{k}{m}\right) \sum_{j=1}^{1} \left(\frac{}{2}\right)^{2(m+1)j} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{2i} C_{2i+2(m+1)j}^{i} \right):
\end{aligned}
\qquad (10)
$$

Using the formula

$$\sum_{i=0}^{1} {}^{i}C^{i}_{2i+r} = \frac{2^{r}}{\sqrt{1 \quad 4}\left(1+\sqrt{1 \quad 4}\right)^{r}}$$

from Graham et al. (1994) (p. 203) and introducing the new discounting coe cient

$$\tilde{} = \frac{}{1+\sqrt{1 \qquad {}^{2}}};$$

(10) rewrites more compactly as

$$\begin{aligned} U_s^C \;=\; & \frac{1}{2(1 \qquad)}\Bigg(\quad B\left(\frac{k}{m}\right) + \tilde{}A\left(\frac{k}{m}\right) \\ & \frac{2\sqrt{1 \qquad {}^{2}}}{} \frac{\tilde{}^{\,m+1}}{1 \qquad \tilde{}^{\,2(m+1)}} \left(\quad B\left(\frac{k}{m}\right) + \tilde{}^{\,m+1}A\left(\frac{k}{m}\right)\right)\Bigg): \end{aligned} \tag{11}$$

In (10), the C_n^l terms are binomial coefficients. Of course, (11) is a a lot easier to understand. Here too, the payoff function (11) contains three parts – ignoring the multiplier, which is a simple discount factor. The first part $\quad B + A$ is the risk-free arbitrage spread (again, the mantra "buy low, sell high"), which could be availed on the unconstrained tree (Figure 3). The second part $(1 \quad \tilde{})A$ is the cost of observing the constraint $0 \quad c$ – having to discard half the binomial tree. This constraint is costly, not only because it implies discarding many trading opportunities (akin to no short-selling), but also because the storage operator may keep buying energy and fail to sell for a long time, in which case discounting renders the operation unprofitable.[8] The last term (the entire second line of (11)) corrects this latter problem; it is the benefit of a capacity constraint, which caps losses from buying energy "forever" – more precisely, for a very long time before selling it. It also includes the benefit of flexibility; that is, having large enough a capacity to divide in sufficiently many steps to not reach the boundary too soon (and be stuck there). As mentioned above, this capacity constraint induces waves in the probability mass, which are also apparent in the last term of (11); these oscillations dampen over time.

The payoff function (11) thus differs from (9) for with proportional bids, the boundaries are

[8] There is path on which the storage operator buys forever, that arises with zero probability almost surely.

never reached (once the lower boundary has been exited). Furthermore, the storage operator does not face the risk associated with a long sequence of negative shocks, in which she keeps buying energy she cannot sell fast enough.[9]

As with the proportional case, we graph this function using the same parameters as before. The dots on the curves correspond to the actual choices that are possible – a fixed quantity, e.g. 0.5. Let $n = 2$, $= 0{:}95$, and $= 0{:}95$. Fig. 6 corresponds to high shocks $a = 0{:}6$ with capacity k moving from 0.15 to 1.15, and Fig. 7 corresponds to low shocks $a = 0{:}2$ where k changes from 0.05 to 0.35.

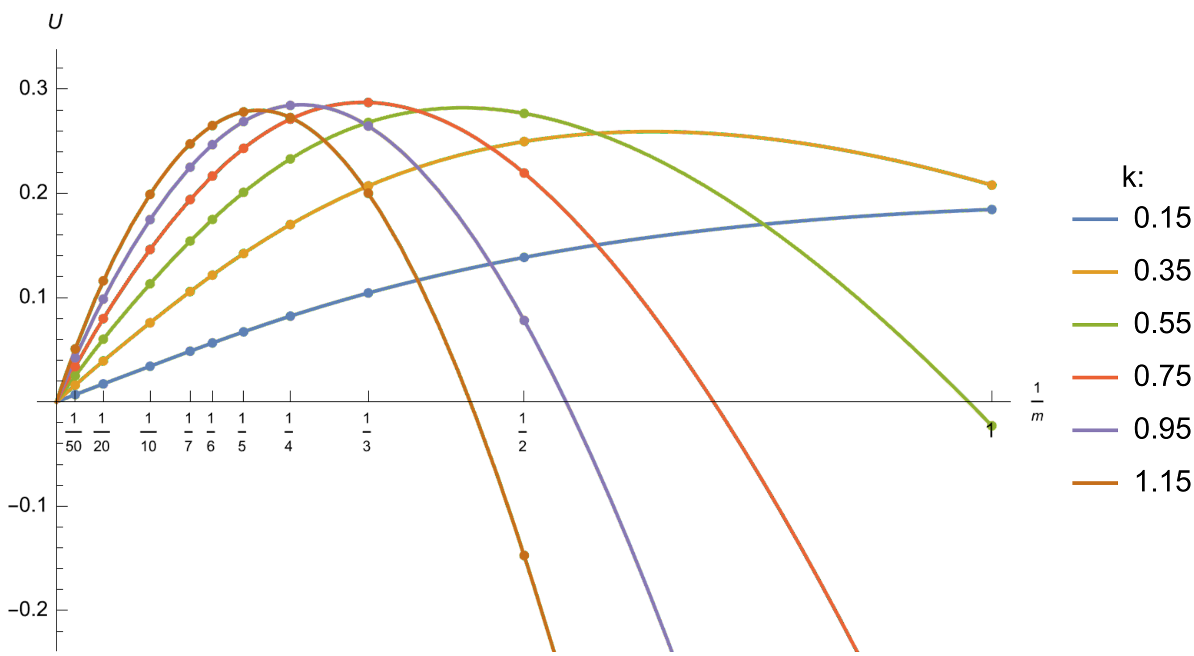

Figure 6: Payoff functions $U_S^C(r)$ for different capacities k when the magnitude of the shock is high enough: $a = 0.6$.

[9] Recall that with proportional bidding, the quantities keep decreasing until a reversal.

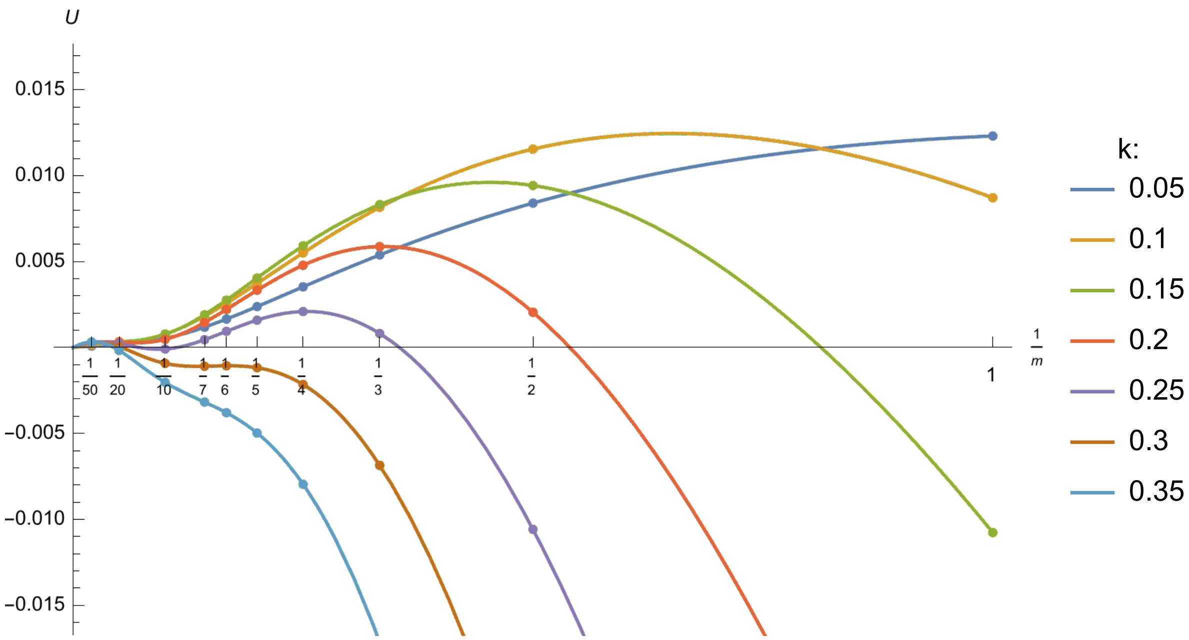

Figure 7: Payoff functions $U_S^C(r)$ for different capacities k when the magnitude of the shock is low: $a = 0.2$.

Overall Figures 6 and 7 complement nicely Figures 1 and 2; they simultaneously enrich them, and confirm the overall message. That is, the optimal quantity choice is interior except for a very small capacity, and rapidly much less than k as capacity increases. In turn the optimal capacity choice is also interior so as to not wipe out the arbitrage spread. Likewise, too small a shock induces too small a spread, which cannot sustain storage.

4.1.3 Direct comparisons

To summarise this section, we lay two different policies on the same graphs for easy comparison (red represents linear bids, and blue represents constant bids). Each picture shows the payoff function under the two heuristics we developed for a given capacity choice k. The first series is concerned with relatively large shocks ($a = 0.6$), and the second one with small shocks ($a = 0.2$).

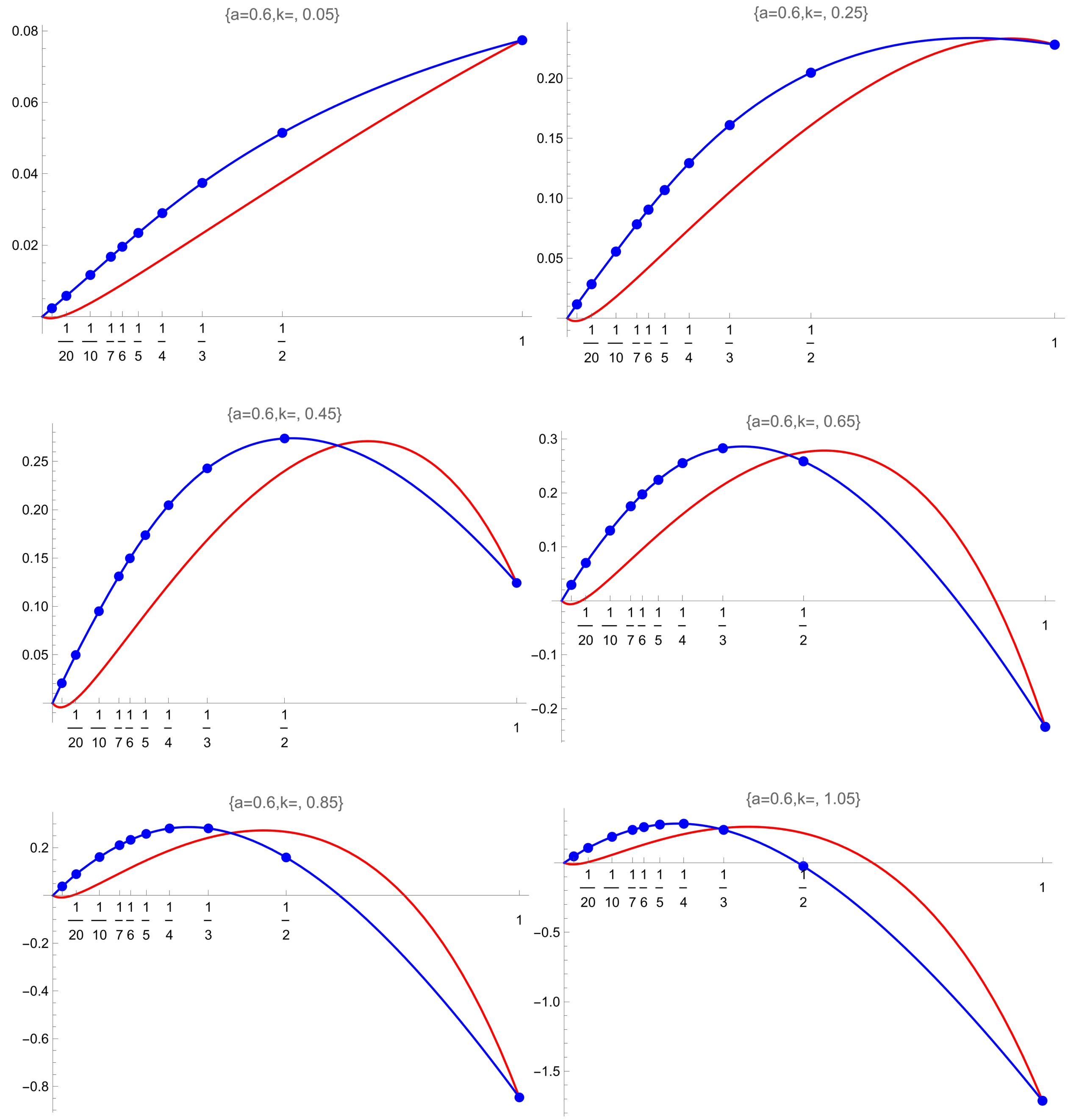

Figure 8: Both proportional and constant bids payoffs in the same graphs for different k and $a = 0.6$.

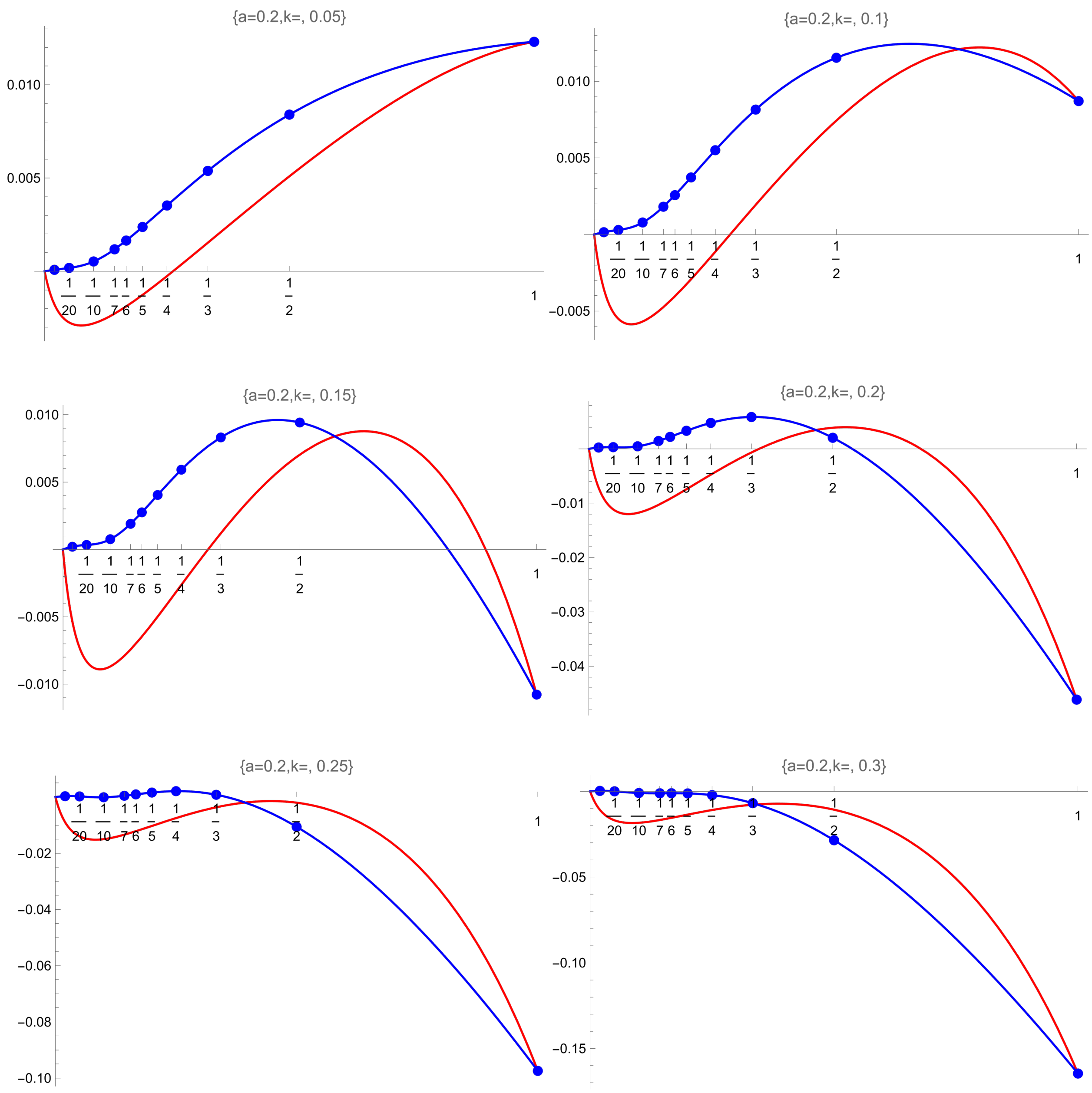

Figure 9: Both proportional and constant bids payoffs in the same graphs for different k and $a = 0.2$.

Some details do differ, and inform us further as to the optimal behaviour of a storage operator. First, it is difficult to rank these two heuristics. While constant quantities do not systematically dominate proportional bidding, the maximum always exceeds – at least weakly – the maximum achieved under the proportional rule, but this may be sensitive to the parameter values we use. We can also see that as capacity k increases, the linear-bid heuristic performs better. This improvement stems from the more flexible nature of the linear-bid heuristic, which never reaches the boundaries for any $r < 1$ and commits the storage operator to either buy ever smaller quantities (in a sequence of negative shocks), or conversely. However, for r large enough, as soon as reversal does occur, the quantity sold at the first positive shock (or sold at the first negative shock), is large. This an approach that is both more prudent and more flexible, which is comparatively better when capacity is large. Indeed, for large shocks and large capacity (the four bottom panels of Figure 8), the maximiser under the proportional rule lies to the right of the maximser under constant bids, and the constant-bid heuristic clear dominates for (many) small steps. A large m delays reaching a boundary at which the storage unit may be stuck. Second, under constant quantities, a storage unit is immune to the (small) losses that accrue under the proportional heuristic when the quantities are very small. Under constant bids, the quantities never become vanishingly small, so the arbitrage revenue is never negligible. We conjecture that a combination of these two approaches is probably even better. It also points to the option of using a more flexible approach to quantity setting; more precisely, relaxing the restriction that $X = k=m$.

Finally we can see that large shocks are completely essential to profitable trading, as see in Figure 9. There a large capacity is of little use.

Of course these differences in behaviour stem from the difference in the stochastic process that is induced by the choice of heuristic. That is, the heuristics interact with the exogenous stochastic process to define an *endogenous* process; this is what defines a stochastic game Shapley (1953). Under a constant-quantity approach, some states become reachable, the probability of any state to be reached changes and the probability mass reflects at the bounds to form waves. Under the proportional heuristic, we already know there is no reflection at the boundaries, but there are oscillations of the probability mass. Also, for r sufficiently large, a reversal is sharp. Going in more details, once the the storage unit leaves the lower boundary, it faces an almost standard binomial tree, in which no new state can emerge. This tree is not quite standard in that once a path is entered – say, through the charge $r(k \quad c)$, it almost surely never crosses the alternative path rc (it may at most once, thanks to the sharp reversal).

4.1.4 Exclusion equilibrium

The emergence of storage is not a foregone conclusion, even absent entry costs. Below we present an equilibrium of the repeated game, in which the storage operator never finds it profitable to incur the cost of the first charge. To make this point we must revert to a simpler structure, in which the storage operator always charges and discharges in full; that is, $r = 1$ or $m = 1$.[10]

Proposition 5. *Assume that*

$$\frac{(1+a)^2(n+1)^2}{4n(1+a^2)+(1+a)^2(n+1)^2} \leqslant \quad \leqslant \frac{2}{1+\frac{2(1+a-k)}{(n+1)(1-a+2k)}}. \tag{12}$$

Then there exists a dynamic Subgame Perfect Nash Equilibrium, such that:

- *in each period, the generators set quantities*
  - $\varepsilon = -a$:
    - $q = \frac{1-a}{2n}$ *if none of the generators deviated in the previous rounds,*
    - $q = \frac{1-a+k}{n+1}$ *if the storage is empty and any of the generators deviated in the previous rounds,*
    - $q = \frac{1-a}{n+1}$ *if the storage is full and any of the generators deviated in the previous rounds;*
  - $\varepsilon = a$:
    - $q = \frac{1+a}{2n}$ *if the storage is empty and none of the generators deviated in the previous rounds,*
    - $q = \frac{1+a}{n+1}$ *if the storage is empty and any of the generators deviated in the previous rounds,*
    - $q = \frac{1+a-k}{n+1}$ *if the storage is full;*

[10] That is not to say this equilibrium does not exist for $r < 1$ or $m > 1$; the equilibrium we present is one of many that can be constructed.

- *the storage does not enter the market if none of the generators have deviated; otherwise, the storage enters the market if*

$$B < \frac{\quad}{2} A: \tag{13}$$

Cumulative consumers' expected payments per period C^0 are

$$C^0 = \frac{1 + a^2}{4}:$$

Expected utilities of the generators U_g^0 and the storage U_s^0 take the following form:

$$U_g^0 = \frac{1 + a^2}{4(1 \quad)n}; \qquad U_s^0 = 0:$$

In this equilibrium, generators collude to the joint-profit maximising quantities and the storage unit never charges. It must deter two kinds of deviations. First, the generators must elect to not deviate; this is supported by the threat of Cournot reversion, which is subgame perfect. Second, the storage operator also prefers not charging, otherwise the punishement reverts to the equilibrium described in the introductory example, which is therefore also subgame perfect. This threat is sufficient because the charging cost is too high compared to what the storage unit can collect once players enter the punishment phase. That is, the strategic effect working through market power supports the equilibrium.

Storage is excluded because it starts empty and must first charge to become active. This is an important consideration that is not studied to its full extent in the works of Andres-Cerezo and Fabra (2022), where exclusion is not considered. If the storage operator could charge anyway, or charge at a preferential rate, then exclusion may not occur. Hence some qualified integration of storage and generation, for selected generation technologies (e.g. solar), may facilitate the emergence and operation of storage. These benefits do not exist in the work of Andres-Cerezo and Fabra (2022).

This result also makes it plain that starting from empty is not just costly for the storage operator; it can be socially costly as well since storage if welfare enhancing. Exclusion may thus be overcome with the help of a small subsidy. Here it is enough to cover the first charge to not only foster storage activity, but also to unravel the collusive equilibrium.

4.2 A richer Markovian structure

The payoff functions we can compute in Section 4.1 feature a cost of uncertainty – see equations (9) and (10). This cost stems for the risk of facing multiple negative shocks, which induce an incentive to charge, but being already fully charged, and conversely.

A glance at the payoff functions (9) and (10) suggests that a sequence of perfectly negatively correlated shocks $-a, a, -a, a, \ldots$ would deliver the highest (and certain) payoff. In this Section we relax the strict independence assumption and investigate the constant-quantity heuristics when shocks follow a non-degenerate Markov chain, and can be made either not persistent or to carry significant persistence. The goal is to better understand the impact of risk in the optimal heuristic of the storage operator.

To this end, we consider a case where the storage buys and sells its capacity in any of one step ($m = 1$), two steps ($m = 2$) or three steps ($m = 3$); richer heuristics, with $m > 3$ are no less interesting but beyond what we can manage. With $m = 2$, for example, there are four possible states of charge that are payoff relevant: an empty unit, a half-full unit after the negative shock, a half-full unit after the positive shock, and finally a full storage unit. There is no need to distinguish the nature of the shock at the boundaries because states 0 and k are accessible only after positive and negative shocks, respectively. Assume now that shocks ε_t form a discrete-time Markov chain:

$$
\begin{aligned}
&Pr\{\varepsilon_0 = a\} = x; && Pr\{\varepsilon_0 = -a\} = 1 - x; \\
&Pr\{\varepsilon_{t+1} = a | \varepsilon_t = a\} = x; && Pr\{\varepsilon_{t+1} = -a | \varepsilon_t = a\} = 1 - x; \\
&Pr\{\varepsilon_{t+1} = a | \varepsilon_t = -a\} = 1 - y; && Pr\{\varepsilon_{t+1} = -a | \varepsilon_t = -a\} = y
\end{aligned}
\tag{14}
$$

for any $t > 0$. We denote the transition matrix by Q and its determinant by d:

$$Q = \begin{pmatrix} x & 1 \quad x \\ 1 \quad y & y \end{pmatrix}; \qquad\qquad d = \det Q = x + y \quad 1;$$

and we let the functions where $A(k{=}m)$ and $B(k{=}m)$ be defined as before.

Proposition 6. *Under conditions laid out below, there exists a dynamic equilibrium, such that*

- *the storage unit buys $k{=}m$ under the negative shock until it reaches capacity k and sells $k{=}m$ under the positive shock until it becomes empty;*
- *in each period, the generators set quantities q according to static Cournot competition and based on the current shock and the state of the storage (full or empty). Namely,*

$$\begin{aligned} q &= \frac{1 \quad a}{n+1} \text{ if storage is full and } " = \quad a; \\ q &= \frac{1 \quad a + k{=}m}{n+1} \text{ if storage is not full and } " = \quad a; \\ q &= \frac{1 + a}{n+1} \text{ if storage is empty and } " = a; \\ q &= \frac{1 + a \quad k{=}m}{n+1} \text{ if storage is not empty and } " = a: \end{aligned}$$

1. *$m = 1$; this equilibrium exists if*

$$B < \frac{(1 \quad y)}{1 \quad y} A: \tag{15}$$

and the expected payo of the storage operator U_s^1 takes the following form:

$$U_s^1 = \frac{1 \quad x}{(1 \quad)(1 \quad d)} (\quad B + \quad (1 \quad y)A + \quad yB):$$

2. $m = 2$; *this equilibrium exists if*

$$B\left(\frac{k}{2}\right) < \frac{(1 \quad y)(1 + \ {}^2 d)}{1 \quad {}^2(1 \quad x + yd)} A\left(\frac{k}{2}\right); \tag{16}$$

and the expected payo of the storage operator U_s^2 *takes the following form:*

$$U_s^2 = \frac{1 \quad x}{(1 \quad)(1 \quad d)}\left(\ B\left(\frac{k}{2}\right) + \ (1 \quad y) A\left(\frac{k}{2}\right) + \ {}^2 y \frac{yB\left(\frac{k}{2}\right) + \ x(1 \quad y)A\left(\frac{k}{2}\right)}{1 \quad {}^2(1 \quad x)(1 \quad y)}\right):$$

3. $m = 3$, *this equilibrium exists if*

$$B\left(\frac{k}{3}\right) < \ (1 \ y)\frac{(1 \quad {}^2(1 \quad x)(1 \quad y))^2 + \ {}^2 xy\,(1 + \ {}^2(xy \quad 2(1 \quad x)(1 \quad y)))}{(1 \quad {}^2(1 \quad x)(1 \quad y))^2 \quad {}^3 y^3 \quad {}^4 xy(1 \quad x)(1 \quad y)} A\left(\frac{k}{3}\right); \tag{17}$$

and the expected payo of the storage operator U_s^3 *reads*

$$\begin{aligned} U_s^3 \ &= \ \frac{1 \quad x}{(1 \quad)(1 \quad d)}\left[\ B\left(\frac{k}{3}\right) + \ (1 \quad y)A\left(\frac{k}{3}\right)\right. \\ &+ \ \ {}^3 y \left.\frac{y^2 B\left(\frac{k}{3}\right) + x(1 \quad y)\,(1 + \ {}^2 d)\,A\left(\frac{k}{3}\right)}{(1 \quad {}^2(1 \quad x)(1 \quad y))^2 \quad {}^4 xy(1 \quad x)(1 \quad y)}\right]: \end{aligned}$$

When there is little persistence, the probability of being stuck at either boundary 0 or k is small; that is, the storage operator is almost guaranteed to reverse direction – for example, sell after charging. Compared to Section 4.1.2, the "waves" are tamed. In turn, this stokes the incentives to charge in the first place, and so on. These rapid cycles reduce the uncertainty, but not the ***volatility***; in fact, certain volatility is best for the storage operator.

More generally, the payoffs $U_s^m; \ m = 1; 2; 3$ include two terms: the first one is $\ B(k{=}m) + \ (1 \quad y)A(k{=}m)$, which is, modulo a multiplier, the discounted payoff from charging and discharging every other period. This is a storage unit operating under perfect foresight. The second term captures the cost of uncertainty, as best-responded to by the storage operator. Here the parameters and the best reply interact richly, as we show next.

Below we plot the payoff functions of the storage operator projected on the dimensions x and y, which denote persistence, and with $x = y$. The red stands for the payoff function when $m = 1$, the blue for $m = 2$ and the green for $m = 3$, all for the constant-quantity heuristic. All other parameters remain unchanged. Low persistence is clearly better in this environment, but we note that in some cases a very high persistence seems to improve payoffs over a moderate persistence: at least the storage operator can sell all its energy with higher probability when full.

When capacity is relatively small (compared to the shock), it is best to charge and discharge in full ($m = 1$) for almost any persistence. The reason is that the storage operator has no (significant) market power, so there is no (significant) price impact. But when capacity increases, we observe more mixed results. First, as persistence increases, flexibility becomes valuable: charging and discharging in two steps and three steps starts dominating. It is better able to cope with the uncertainty of the shocks. Second, with a large(r) capacity, restraint also pays off: charging and discharging in two steps (blue) dominates one step (red) for any persistence; and three steps (green) dominates both. It is best to not use the capacity in full at any point in time because of the market power effect, and this is exactly what $m = 3$ delivers. These conclusions are replicated, but even starker, when shocks and capacity are even smaller. Then, in some cases, the three-step strategy is the only one that can deliver any positive surplus.

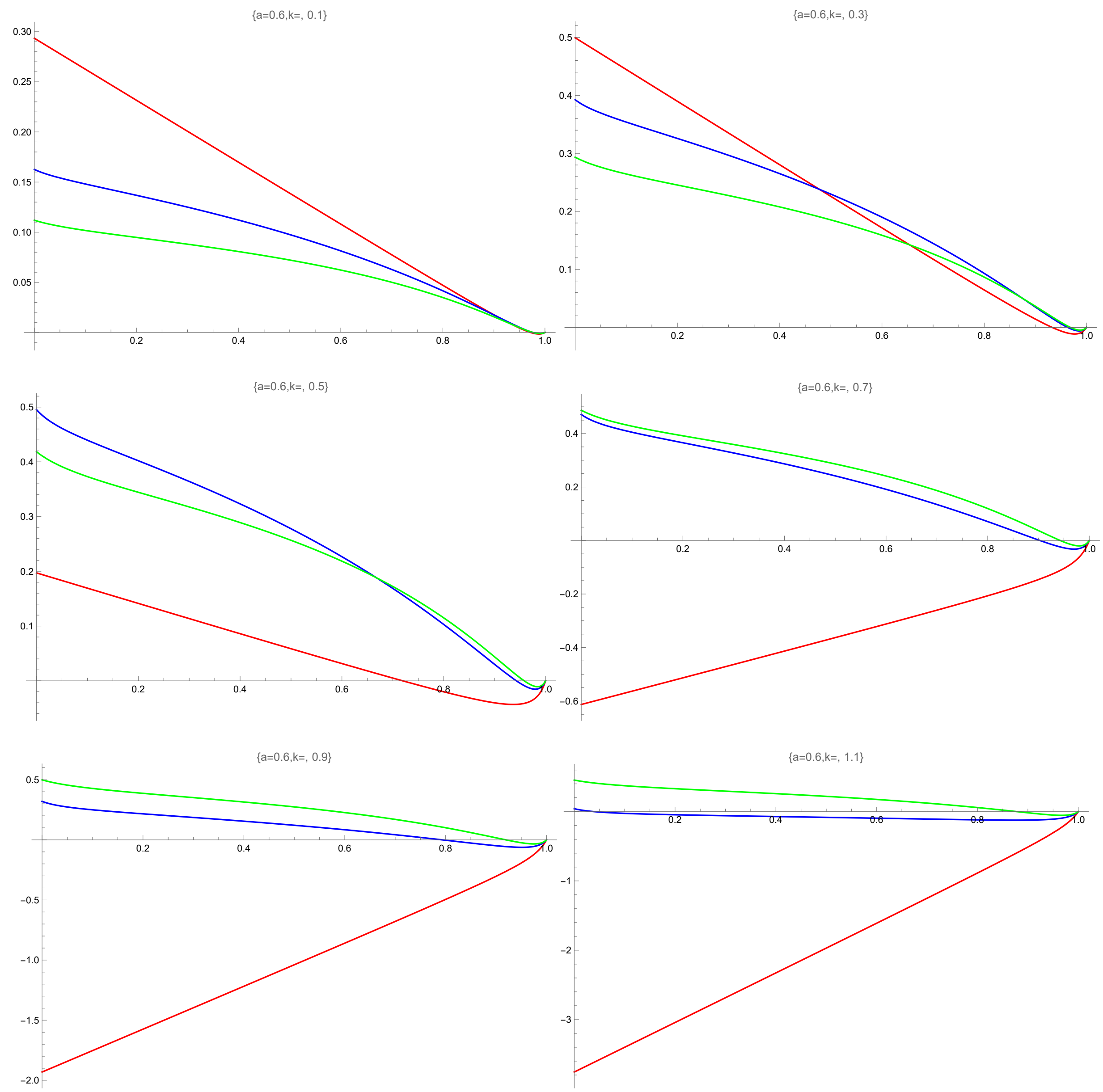

Figure 10: Payoffs under Markov shocks for divisible (green for $k/3$ and blue for $k/2$) and indivisible (red) capacities in the same graphs for different k and $a = 0.6$.

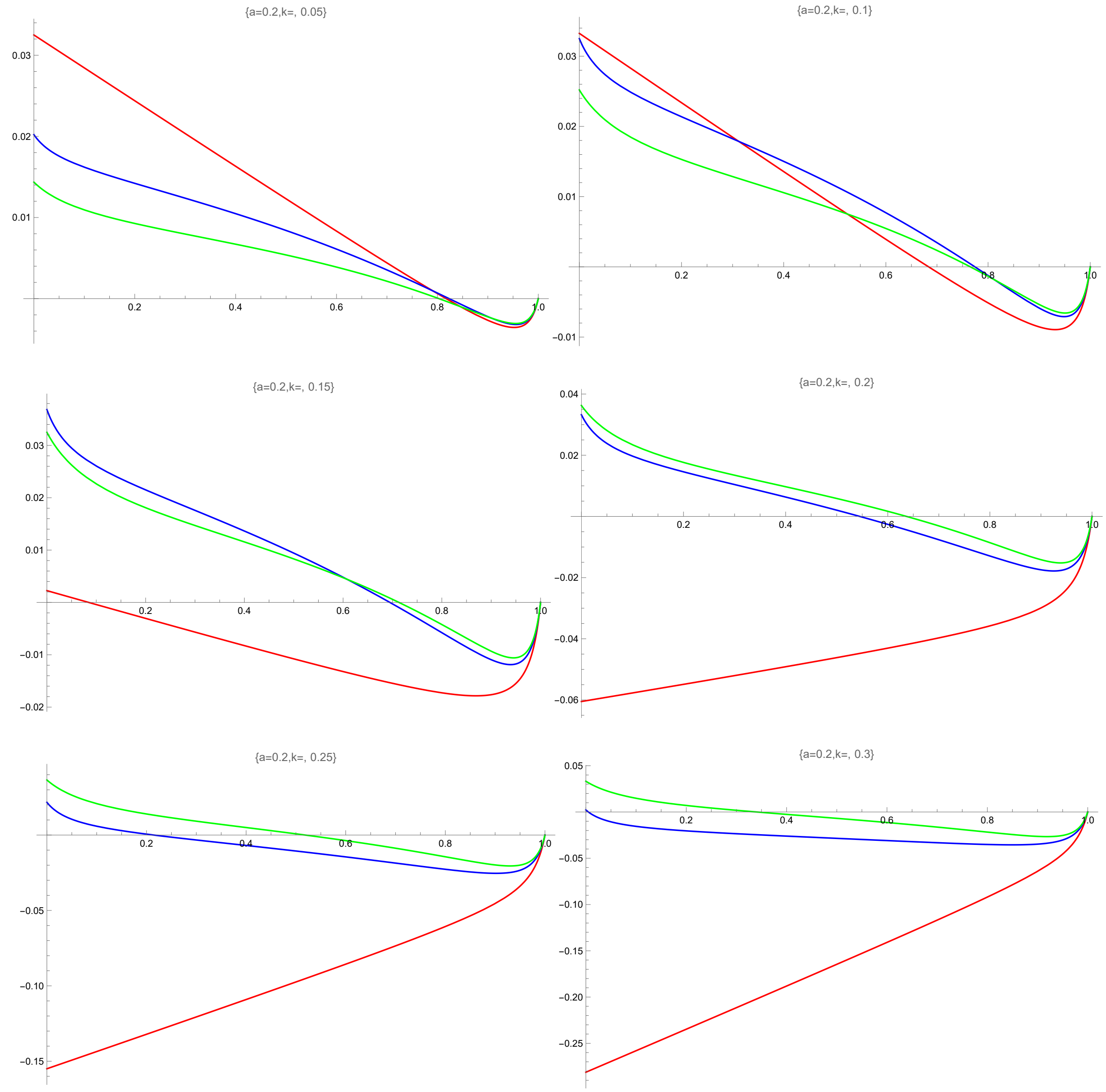

Figure 11: Payoffs under Markov shocks for divisible (green for $k/3$ and blue for $k/2$) and indivisible (red) capacities in the same graphs for different k and $a = 0.2$.

With this simple structure it is difficult to speak of the impact of high persistence in greater detail. With a lot of persistence in shock (x ! 1 or y ! 1), the storage operator can spend a lot of time at either boundary (0 or k); if that is the case, one can conjecture she would like to buy or sell over many periods (so, m be large). We cannot treat this case, and aside from a strict numerical treatment, there is no hope of doing so because even a computer cannot find the eigenvalues of the matrices of interest.

5 Conclusion

In this paper we study the dynamic trading of electricity based on storage. This is an important step to understand the economics of electricity storage, and to tackle the ambitious question of market design with storage. We limit ourselves to a simple Cournot environment with stochastic shocks that follow a Markov chain; there is no change in the mean demand. This environment features market power and strategic behaviour, in departure to much of the literature on (other forms of) storage.

Even then, the analysis of such a simple problem is very demanding. To make progress, we must confine ourselves to studying simple heuristics, which allows us to derive almost explicit forms for the long-horizon payoffs of the storage unit. We uncover three competing forces that a storage operator must balance: market power (given capacity), capacity choice (for an optimal heuristic) and continuation. We also find that risk is costly to the storage unit; that risk is the probability to be stuck either empty or full and being unable to either buy or sell.

There is still a tremendous amount of work to do to really understand the economics of storage. In this model there are no binding capacity constraints on generators, and no change in the mean demand over time. Both features are central to electricity markets but difficult to incorporate in a model of dynamic trading.

A Appendix

A.1 Proof of the Introductory example

According to (4) and (2), the storage operator buys k units under price $p = (1 \quad a + k)=(n + 1)$ and sells k units under price $p = (1 + a \quad k)=(n + 1)$. The probability that the storage observes the first positive shock after i periods is $(1=2)^{i \quad 1} \quad (1=2)$. Thus, the total discounting of waiting for the positive shock after recharging is equal to

$$\frac{1}{2} \quad + \frac{1}{4} \quad {}^2 + \quad + \frac{1}{2^i} \quad {}^i + \quad = \frac{}{2 \quad}:$$

Hence, entering the market is profitable for the storage operator if $B + \frac{}{2 \quad} A > 0$.

Four possible deviations of the storage unit should be considered. All other deviations are just compositions of those four.

- The unit is full but deviates by not selling under the positive shock. Then there may be only loss comparing to the default strategy. Indeed, nothing changes on the market except the future profits to be discounted by .

- The unit is empty and deviates by not buying under the negative shock. Also, no gains here.

- The unit is full and deviates by selling under the negative shock. In this situation, the quantities supplied by the generators are $q = (1 \quad a)=(n + 1)$. The resulting price after the deviation is

  $$p = 1 \quad a \quad k \quad n\frac{1 \quad a}{n + 1} = \frac{1 \quad a}{n + 1} \quad k:$$

  To make this deviation profitable, the storage operator must gain more than if it waits for the positive shock and sells in that period:

  $$\left(\frac{1 \quad a}{n + 1} \quad k\right) k > \frac{}{2 \quad} \frac{1 + a \quad k}{n + 1} k;$$

  which is impossible when $B < \frac{}{2 \quad} A$.

- The unit is empty and deviates by buying under the positive shock. Here we have $q = (1+a)=(n+1)$, and the resulting price after the deviation is $p = (1+a)=(n+1)+k$. The profits after selling the purchased energy are:

$$
\begin{aligned}
& \left(\frac{1+a}{n+1}+k\right)k + \frac{\quad}{2}\frac{1+a \qquad k}{n+1} \; k \\
= \quad & \frac{k}{n+1}\left(\left(1 \quad \frac{\quad}{2}\right)(1+a)+\left(\frac{\quad}{2}\;{}^{2}+n+1\right)k\right) < 0:
\end{aligned}
$$

  There are no gains from this deviation.

Ruling out deviations of the generators is simple. In each round, we have a static Cournot equilibrium for all the participants. Thus, any change of the equilibrium quantity in round t leads to decreasing the payoffs in that round and, thus, decreasing the overall payoffs. Indeed, the stage-game Cournot equilibrium is an equilibrium also in the long-horizon game.

To find the expected payoff of the storage unit, let's introduce value function $V_t(i); \; i \; 2 \; f1;0g$. $V_t(i)$ is the total expected payoff of the storage unit from moment t if the current state is full ($i = 1$) or empty ($i = 0$). We have a system of recursive equations:

$$
\begin{cases}
V_t(1) = \frac{1}{2} \;\; (A + \;\; V_{t+1}(0)) + \frac{1}{2} \qquad V_{t+1}(1); \\
V_t(0) = \frac{1}{2} \qquad V_{t+1}(0) + \frac{1}{2} \;\; (\;\; B + \qquad V_{t+1}(1)):
\end{cases}
$$

It can be rewritten in a matrix form

$$
V_t = P + \qquad Q \;\; V_{t+1}; \tag{18}
$$

where

$$
V_t = \begin{pmatrix} V_t(1) \\ V_t(0) \end{pmatrix}; \qquad P = \frac{1}{2}\begin{pmatrix} A \\ \;\; B \end{pmatrix}; \qquad Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}:
$$

Note that $Q^2 = Q$. For $\; < 1$, we can find from (18) that

$$V_0 = P + \sum_{i=1}^{t} \; {}^{i}Q^{i} \;\; P + \;\; {}^{t+1}Q^{t+1} \;\; V_{t+1} =$$

$$= P + \frac{(1 \quad {}^{t})}{1 \qquad} \;\; Q \;\; P + \;\; {}^{t+1} \;\; Q \;\; V_{t+1} \underset{t \,!\, 1}{!} \; P + \frac{\qquad}{1 \qquad} \;\; Q \;\; P = \frac{1}{4} \begin{pmatrix} \frac{2}{1}\!\!\!-\!A & \frac{}{1}\!\!-\!B \\ \frac{}{1}\!\!-\!A & \frac{2}{1}\!\!-\!B \end{pmatrix} : \tag{19}$$

The lower term is exactly U_s.

To find the expected payoff of the generators, let's introduce value function $W_t(i)$, $i \; 2$ $f1; 0g$. $W_t(i)$ is the total expected payoff of a generator from moment t if the current state of the storage is full ($i = 1$) or empty ($i = 0$). We have a system of recursive equations:

$$\begin{cases} W_t(1) = \frac{1}{2} \;\; (G_{10} + \;\; W_{t+1}(1)) + \frac{1}{2} \;\; (G_{11} + \quad W_{t+1}(0)) \, ; \\ W_t(0) = \frac{1}{2} \;\; (G_{00} + \quad W_{t+1}(1)) + \frac{1}{2} \;\; (G_{01} + \quad W_{t+1}(0)) \, : \end{cases}$$

It can be rewritten in a matrix form

$$W_t = R + \quad Q \;\; W_{t+1};$$

where

$$W_t = \begin{pmatrix} W_t(1) \\ W_t(0) \end{pmatrix}; \qquad R = \frac{1}{2} \begin{pmatrix} G_{10} + G_{11} \\ G_{00} + G_{01} \end{pmatrix}; \qquad Q = Q^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} :$$

Using the same algebra as for $V(t)$ earlier, we obtain

$$W_0 \underset{t \,!\, 1}{!} \; R + \frac{\qquad}{1 \qquad} \;\; Q \;\; R = \begin{pmatrix} \frac{1}{2}(G_{10} + G_{11}) + \frac{\qquad}{4(1 \quad)}(G_{10} + G_{11} + G_{00} + G_{01}) \\ \frac{1}{2}(G_{00} + G_{01}) + \frac{\qquad}{4(1 \quad)}(G_{10} + G_{11} + G_{00} + G_{01}) \end{pmatrix} :$$

The lower term is exactly U_g.

To find cumulative consumers' expected payments per period, notice first that each of four states – a combination of a full/empty storage and a positive/negative shock – has the same probability 1=4. Table 1 summarizes all the consumers' payments under each of those events.

shock n storage	empty	full
negative	$\frac{1 \quad a+k}{n+1} \left(n \frac{1 \quad a+k}{n+1} \quad k \right)$	$\frac{1 \quad a}{n+1} \; n \frac{1 \quad a}{n+1}$
positive	$\frac{1+a}{n+1} \; n \frac{1+a}{n+1}$	$\frac{1+a \quad k}{n+1} \left(n \frac{1+a \quad k}{n+1} + k \right)$

Table 1

Summing up all the cells with weight 1=4 for each, we get C_1.

A.2 Proofs of the Propositions

A.2.1 Proof of Proposition 3

Since under proportional bids, we never reach upper limit k and never reach lower limit 0 again after starting there, the system of equations (8) takes the following form:

$$\begin{cases} V(0) & = \frac{1}{2} \left(\frac{1 \quad a+rk}{n+1} \quad rk + \quad V(rk) \right); \\ V(c) & = \frac{1}{2} \left(\frac{1 \quad a+r(k \quad c)}{n+1} \quad r(k \quad c) + \quad V(c + r(k \quad c)) \right) \\ & + \frac{1}{2} \left(\frac{1+a \quad rc}{n+1} \quad rc + \quad V((1 \quad r)c) \right): \end{cases}$$

We need to find $V(0) = U_s^P$. Let's enumerate all c, $b(c)$, and $a(c)$ in order of their appearance when we expand our equation for $V(0)$. Namely, in the first period we have

$$V(0) = \frac{1}{2} \quad V(0) + \frac{1}{2} \left(\frac{1 \quad a+rk}{n+1} \quad rk + \quad V(rk) \right) = \quad \frac{1}{2} B(rk) + \frac{}{2} \left(V(0) + V(rk) \right);$$

so we define $c^0 = 0$, $b^0 = r(k \; c^0) = rk$, $c^1 = c^0 + b^0 = rk$. In the second round, we get

$$\begin{aligned} V(0) = \; & \frac{1}{2}B(rk) + \frac{}{2}\left(\frac{1}{2} \; V(0) + \frac{1}{2}(\; B(rk) + \; V(rk)) \right. \\ & \left. + \frac{1}{2}(\; B(r(k \; rk)) + \; V(rk + r(k \; rk))) + \frac{1}{2}(A(r \; rk) + \; V((1 \; r)rk))\right) \\ = \; & \frac{1}{2}B(rk) + \frac{}{4}\Big(\; B(rk) \; B(r(1 \; r)k) + A(r^2k)\Big) \\ & + \frac{^2}{4}\Big(V(0) + V(rk) + V(r(2 \; r)k) + V(r(1 \; r)k)\Big): \end{aligned}$$

Thus,

$$b^1 = r(k \; c^1) = r(1 \; r)k; \qquad a^1 = rc^1 = r^2k;$$

$$c^2 = c^1 + b^1 = r(2 \; r)k; \qquad c^3 = c^1 \; a^1 = r(1 \; r)k:$$

In the third round, we get

$$\begin{aligned} V(0) = \; & \frac{1}{2}B(b^0) + \frac{}{4}\big(\; B(b^0) \; B(b^1) + A(a^1)\big) \\ & + \frac{^2}{8}\big(\; B(b^0) \; B(b^1) \; B(b^2) \; B(b^3) + A(a^1) + A(a^2) + A(a^3)\big) + \frac{^3}{8}\sum_{i=0}^{7} V(c^i); \end{aligned}$$

where

$$b^2 = r(k \; c^2) = r(1 \; 2r \; r^2)k; \qquad b^3 = r(k \; c^3) = r(1 \; r + r^2)k;$$

$$a^2 = rc^2 = r^2(2 \; r)k; \qquad a^3 = rc^3 = r^2(1 \; r)k;$$

$$c^4 = c^2 + b^2 = r(3 \; 3r \; r^2)k; \qquad c^5 = c^3 + b^3 = r(2 \; 2r + r^2)k;$$

$$c^6 = c^2 \; a^2 = r(1 \; r)(2 \; r)k; \qquad c^7 = c^3 \; a^3 = r(1 \; r)^2k:$$

Finally, in round t,

$$V(0) = \frac{1}{2}B(b^0) + \frac{\ }{4}\left(\ B(b^0) \ B(b^1) + A(a^1)\right) + \frac{\ ^2}{8}\left(\ \sum_{i=0}^{3} B(b^i) + \sum_{i=1}^{3} A(a^i)\right) + $$
$$+ \frac{\ ^{t-1}}{2^t}\left(\ \sum_{i=0}^{2^{t-1}-1} B(b^i) + \sum_{i=1}^{2^{t-1}-1} A(a^i)\right) + \frac{\ ^t}{2^t}\sum_{i=0}^{2^t-1} V(c^i),$$

where b^i, a^i, and c^i can be found recursively. Continuing this process infinitely and noticing that

$$\frac{\ ^t}{2^t}\sum_{i=0}^{2^t-1} V(c^i) \leqslant \ ^t V(\max_i c^i) \xrightarrow[t\to\infty]{} 0,$$

we obtain:

$$V(0) \xrightarrow[t\to\infty]{} \frac{1}{2}B(b^0) + \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{\ }{2}\right)^j \left(\ \sum_{i=0}^{2^j-1} B(b^i) + \sum_{i=1}^{2^j-1} A(a^i)\right). \tag{20}$$

Let

$$G(t) = \sum_{i=0}^{2^t-1} c^i, \qquad H(t) = \sum_{i=0}^{2^t-1} \left(c^i\right)^2.$$

We can get a recursive equation for $G(t)$:

$$G(t) = \sum_{i=0}^{2^{t-1}-1} (c^i + b^i) + \sum_{i=1}^{2^{t-1}-1} (c^i - a^i)$$
$$= \sum_{i=0}^{2^{t-1}-1} (c^i + r(k - c^i)) + \sum_{i=0}^{2^{t-1}-1} (c^i - rc^i) = rk2^{t-1} + 2(1-r)G(t-1),$$

which implies

$$G(t) = rk2^{t-1} + 2(1-r)\left(rk2^{t-2} + 2(1-r)G(t-2)\right) = \ldots$$
$$= rk\sum_{i=0}^{t-1} 2^i(1-r)^i 2^{t-1-i} + 2^t(1-r)^t G(0) = k2^{t-1}\left(1-(1-r)^t\right). \tag{21}$$

The same technique works for $H(t)$:

$$H(t) = \sum_{i=0}^{2^{t-1}-1} (c^i + b^i)^2 + \sum_{i=1}^{2^{t-1}-1} (c^i - a^i)^2 = \sum_{i=0}^{2^{t-1}-1} (rk + (1-r)c^i)^2 + \sum_{i=0}^{2^{t-1}-1} ((1-r)c^i)^2$$
$$= r^2k^2 2^{t-1} + 2r(1-r)kG(t-1) + (1-r)^2 H(t-1) + (1-r)^2 H(t-1)$$
$$= rk^2 2^{t-1}\left(1-(1-r)^t\right) + 2(1-r)^2 H(t-1),$$

and

$$H(t) = rk^2 2^{t-1}\left(1-(1-r)^t\right) + 2(1-r)^2\Big(rk^2 2^{t-2}\left(1-(1-r)^{t-1}\right) + 2(1-r)^2 H(t-2)\Big)$$
$$= \dots = rk^2 \sum_{i=0}^{t-1} 2^i (1-r)^{2i} 2^{t-i-1}\left(1-(1-r)^{t-i}\right) + 2^t (1-r)^{2t} H(0)$$
$$= k^2 2^{t-1} \frac{(1-(1-r)^t)(1-(1-r)^{t+1})}{2-r}. \tag{22}$$

From (21), we can find expressions for sums of b^i and a^i:

$$\sum_{i=0}^{2^t-1} b^i = \sum_{i=0}^{2^t-1} r(k-c^i) = rk2^t - rG(t) = rk2^{t-1}\left(1+(1-r)^t\right);$$
$$\sum_{i=0}^{2^t-1} a^i = \sum_{i=0}^{2^t-1} rc^i = rG(t) = rk2^{t-1}\left(1-(1-r)^t\right).$$

From (21) and (22), we can find expressions for sums of squares of b^i and a^i:

$$\sum_{i=0}^{2^t-1} \left(b^i\right)^2 = \sum_{i=0}^{2^t-1} r^2(k-c^i)^2$$
$$= r^2k^2 2^t - 2r^2kG(t) + r^2H(t) = r^2k^2 2^{t-1}\left(\frac{1+(1-r)^{2t+1}}{2-r} + (1-r)^t\right);$$
$$\sum_{i=0}^{2^t-1} \left(a^i\right)^2 = \sum_{i=0}^{2^t-1} r^2 \left(c^i\right)^2 = r^2 H(t) = r^2k^2 2^{t-1}\left(\frac{1+(1-r)^{2t+1}}{2-r} - (1-r)^t\right).$$

Now we are ready to calculate sums of $B(b(i))$ and $A(a(i))$ and get the final formula for

$V(0)$. From (20), we have

$$
\begin{aligned}
V(0) = \;& \frac{1}{2}B(rk) + \frac{1}{2}\sum_{j=1}^{1}\left(\frac{\;}{2}\right)^{j}\left(\;\; \sum_{i=0}^{2^j \;\; 1}\frac{1 \;\;\; a+b^i}{n+1}b^i + \sum_{i=1}^{2^j \;\; 1}\frac{1+a \;\;\;\; a^i}{n+1}\;\; a^i\right) \\
= \;& \frac{1}{2}\frac{1 \;\;\; a+rk}{n+1}rk + \frac{1}{2(n+1)}\sum_{j=1}^{1}\left(\frac{\;}{2}\right)^{j}\Big(\;\; (1 \;\;\; a)rk2^{j \;\; 1}\left(1+(1 \;\;\; r)^j\right) \\
& r^2k^2 2^{j \;\; 1}\left(\frac{1+(1 \;\;\; r)^{2j+1}}{2 \;\;\; r} + (1 \;\;\; r)^j\right) + (1+a)\;\; rk2^{j \;\; 1}\left(1 \;\;\; (1 \;\;\; r)^j\right) \\
& {}^2 r^2k^2 2^{j \;\; 1}\left(\frac{1+(1 \;\;\; r)^{2j+1}}{2 \;\;\; r} \;\;\; (1 \;\;\; r)^j\right)\Big) \\
= \;& \frac{rk}{4(n+1)(1 \;\;\;\;\;)(1 \;\;\; (1 \;\;\; r) \;\;)}\Big(r \;\;\; ((1+a) \;\;\;\; (1 \;\;\; a)) \;\;\; 2(1 \;\;\;\;\;)(1 \;\;\; a+rk) \\
& rk\frac{r^2(1+ \;\;\; {}^2)}{1 \;\;\; (1 \;\;\; r)^2}\Big):
\end{aligned}
$$

The last expression is exactly formula (9).

A.2.2 Proof of Proposition 4

System of equations (8) takes the following form:

$$
\begin{cases}
V_t\left(\frac{ik}{m}\right) = \frac{1}{2}\left(A\left(\frac{k}{m}\right) + \;\; V_{t+1}\left(\frac{(i \;\;\; 1)k}{m}\right) \;\;\; B\left(\frac{k}{m}\right) + \;\; V_{t+1}\left(\frac{(i+1)k}{m}\right)\right); \\
V_t(0) = \frac{1}{2}\left(\;\; V_{t+1}(0) \;\;\; B\left(\frac{k}{m}\right) + \;\; V_{t+1}\left(\frac{k}{m}\right)\right); \\
V_t(k) = \frac{1}{2}\left(A\left(\frac{k}{m}\right) + \;\; V_{t+1}\left(\frac{(m \;\;\; 1)k}{m}\right) + \;\; V_{t+1}(k)\right)
\end{cases} \tag{23}
$$

for all $1 \; 6 \; i \; 6 \; m \;\;\; 1$. We are interested in coefficients c_t^i in front of value functions $V_t(ik{=}m)$ for each particular $t > 0$ and $0 \; 6 \; i \; 6 \; m$, such that

$$
V_0(0) = F_{t \;\; 1}\left(\;\; ; A\left(\frac{k}{m}\right); B\left(\frac{k}{m}\right)\right) + \left(\frac{\;}{2}\right)^{t}\sum_{i=0}^{m} c_t^i V_t\left(\frac{ik}{m}\right): \tag{24}
$$

Note that $\sum_i c_t^i = 2^t$ for any t.

In each period t, the storage buys energy with probability $1{=}2$. This cannot be done only if the storage has reached its full capacity. Also, in period t the storage sells energy with probability $1/2$ if it's not empty. Thus, the overall expected earnings of the storage up to period t can be described by the following expression:

$$F_t = \sum_{i=0}^{t} \left(\frac{\beta}{2}\right)^i \frac{1}{2} \left((2^i - c_i^0) A\left(\frac{k}{m}\right) - (2^i - c_i^m) B\left(\frac{k}{m}\right) \right).$$

Indeed, $c_t^0/2^t$ and $c_t^m/2^t$ are the probabilities of the storage to be correspondingly empty or full at t, according to (24).

Since $\sum_i c_t^i = 2^t$, $\beta < 1$, and V_t is a nondecreasing function, the last term in (24) goes to zero if $t \to \infty$:

$$\left(\frac{\beta}{2}\right)^t \sum_{i=0}^{m} c_t^i V_t\left(\frac{ik}{m}\right) \leq \left(\frac{\beta}{2}\right)^t \sum_{i=0}^{m} c_t^i V_t(k) = \beta^t V_t(k) \xrightarrow[t \to \infty]{} 0.$$

Then the expected payoff function takes the following form:

$$U_s^C = V_0(0) = \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^i \frac{1}{2} \left((2^i - c_i^0) A\left(\frac{k}{m}\right) - (2^i - c_i^m) B\left(\frac{k}{m}\right) \right). \tag{25}$$

We need to find c_i^0 and c_i^m. From (23), we can see that

$$c_{t+1}^0 = c_t^0 + c_t^1; \qquad c_{t+1}^i = c_t^{i-1} + c_t^{i+1} \quad (1 \leq i \leq m-1); \qquad c_{t+1}^m = c_t^{m-1} + c_t^m.$$

Thus, we have a modified version of Pascal's triangle. Let's see if we can express c_t^i in terms of binomial coefficients $C_n^k = n!/(k!(n-k)!)$. The two main properties of C_n^k we are going to use are

$$C_n^k = C_n^{n-k}; \qquad\qquad C_n^k + C_n^{k+1} = C_{n+1}^{k+1}.$$

We start with $c_0^0 = 1 = C_0^0$ (and all other $c_0^i = 0$, $i > 1$). In period 1, we have $c_1^1 = c_1^0 = 1 = C_1^0$ with all other c_1^i equal to zero. Period 2 delivers $c_2^2 = c_2^1 = 1 = C_2^0$ and $c_2^0 = c_1^1 + c_0^1 = C_1^0 + C_1^0 = C_1^1 + C_1^0 = C_2^1$, with all remaining c_1^i equal to zero. In period 3, we have $c_3^3 = c_3^2 = 1 = C_3^0$, $c_3^1 = c_2^2 + c_2^0 = C_2^0 + C_2^1 = C_3^1$, and $c_3^0 = c_2^1 + c_2^0 = C_2^0 + C_2^1 = C_3^1$, with all other c_1^i equal to zero. We get rid of all the zeroes by period m with $c_m^i = C_m^{\lfloor (m-i)/2 \rfloor}$ (here, $\lfloor x \rfloor$ is the largest integer which is less than or equal to x). This process is summarized in Table 2.

state of charge i \ time t	0	1	2	3	4	. . .	m − 2	m − 1	m
m	0	0	.	.	.	. . .	0	0	C_m^0
m − 1	0	0	.	.	.	. . .	0	C_{m-1}^0	C_m^0
m − 2	0	0	.	.	.	. . .	C_{m-2}^0	C_{m-1}^0	C_m^1
⋮	⋮					⋮			⋮
2	0	0	C_2^0	C_3^0	C_4^1	. . .	$C_{m-2}^{\lfloor (m-4)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-3)/2 \rfloor}$	$C_m^{\lfloor (m-2)/2 \rfloor}$
1	0	C_1^0	C_2^0	C_3^1	C_4^1	. . .	$C_{m-2}^{\lfloor (m-3)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-2)/2 \rfloor}$	$C_m^{\lfloor (m-1)/2 \rfloor}$
0	C_0^0	C_1^0	C_2^1	C_3^1	C_4^2	. . .	$C_{m-2}^{\lfloor (m-2)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-1)/2 \rfloor}$	$C_m^{\lfloor m/2 \rfloor}$
c_t^i					time t				

Table 2: The first $m+1$ steps of evolving c_t^i

However, after period m we cannot go up anymore. Instead, all the extra mass we accumulate goes down step by step. Namely, in period $m+1$ we still have $c_{m+1}^i = C_{m+1}^{\lfloor (m+1-i)/2 \rfloor}$ for all $0 \leq i \leq m-1$, but for $i = m$ we now have $c_{m+1}^m = C_{m+1}^0 + C_{m+1}^0$. In period $m+2$, we still have $c_{m+2}^i = C_{m+2}^{\lfloor (m+2-i)/2 \rfloor}$, but only for $0 \leq i \leq m-2$. For $i = m$ and $i = m-1$, we have $c_{m+2}^m = c_{m+2}^{m-1} = C_{m+2}^1 + C_{m+2}^0$, etc. Finally, in period $2m+1$, we have

$$c_{2m+1}^i = C_{2m+1}^{\lfloor \frac{2m+1-i}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{i}{2} \rfloor} \qquad 0 \leq i \leq m.$$

See Table 3 for the entire picture.

m	C_m^0	$C_{m+1}^0+C_{m+1}^0$	$C_{m+2}^1+C_{m+2}^0$	...	$C_{2m-1}^{\lfloor\frac{m-1}{2}\rfloor}+C_{2m-1}^{\lfloor\frac{m-2}{2}\rfloor}$	$C_{2m}^{\lfloor\frac{m}{2}\rfloor}+C_{2m}^{\lfloor\frac{m-1}{2}\rfloor}$	$C_{2m+1}^{\lfloor\frac{m+1}{2}\rfloor}+C_{2m+1}^{\lfloor\frac{m}{2}\rfloor}$
m−1	C_m^0	C_m^1	$C_{m+2}^1+C_{m+2}^0$	...	$C_{2m-1}^{\lfloor\frac{m}{2}\rfloor}+C_{2m-1}^{\lfloor\frac{m-3}{2}\rfloor}$	$C_{2m}^{\lfloor\frac{m+1}{2}\rfloor}+C_{2m}^{\lfloor\frac{m-2}{2}\rfloor}$	$C_{2m+1}^{\lfloor\frac{m+2}{2}\rfloor}+C_{2m+1}^{\lfloor\frac{m-1}{2}\rfloor}$
m−2	C_m^1	C_{m+1}^1	C_{m+2}^2	...	$C_{2m-1}^{\lfloor\frac{m+1}{2}\rfloor}+C_{2m-1}^{\lfloor\frac{m-4}{2}\rfloor}$	$C_{2m}^{\lfloor\frac{m+2}{2}\rfloor}+C_{2m}^{\lfloor\frac{m-3}{2}\rfloor}$	$C_{2m+1}^{\lfloor\frac{m+3}{2}\rfloor}+C_{2m+1}^{\lfloor\frac{m-2}{2}\rfloor}$
....	⋮			⋮			⋮
2	$C_m^{\lfloor\frac{m-2}{2}\rfloor}$	$C_{m+1}^{\lfloor\frac{m-1}{2}\rfloor}$	$C_{m+2}^{\lfloor\frac{m}{2}\rfloor}$	...	$C_{2m-1}^{m-2}+C_{2m-1}^0$	$C_{2m}^{m-1}+C_{2m}^0$	$C_{2m+1}^{m-1}+C_{2m+1}^1$
1	$C_m^{\lfloor\frac{m-1}{2}\rfloor}$	$C_{m+1}^{\lfloor\frac{m}{2}\rfloor}$	$C_{m+2}^{\lfloor\frac{m+1}{2}\rfloor}$	...	C_{2m-1}^{m-1}	$C_{2m}^{m-1}+C_{2m}^0$	$C_{2m+1}^m+C_{2m+1}^0$
0	$C_m^{\lfloor\frac{m}{2}\rfloor}$	$C_{m+1}^{\lfloor\frac{m+1}{2}\rfloor}$	$C_{m+2}^{\lfloor\frac{m+2}{2}\rfloor}$	...	C_{2m-1}^{m-1}	C_{2m}^m	$C_{2m+1}^m+C_{2m+1}^0$
i/t	**m**	**m+1**	**m+2**	...	**2m−1**	**2m**	**2m+1**

Table 3: The second $m+1$ steps of evolving c_t^i

Now the excess mass reached the lower boundary again. Since we cannot go down anymore, this mass has to spread up again and add one more binomial coefficient as a summand to all c_t^i starting from $t = 2m + 2$ and until $t = 3m + 2$. This process continues infinitely. We can now derive c_i^0 and c_i^m:

$$c_i^0 = \begin{cases} C_i^{b\frac{i}{2}c} & \text{if} \quad 0 \; 6 \; i \; 6 \; 2m; \\ C_i^{b\frac{i}{2}c} + C_i^{b\frac{i \quad 2m \quad 1}{2}c} & \text{if} \quad i = 2m+1; \\ C_i^{b\frac{i}{2}c} + C_i^{b\frac{i \quad 2m \quad 1}{2}c} + C_i^{b\frac{i \quad 2m \quad 2}{2}c} & \text{if} \quad 2m+2 \; 6 \; i \; 6 \; 2m+2+2m; \\ C_i^{b\frac{i}{2}c} + C_i^{b\frac{i \quad 2m \quad 1}{2}c} + C_i^{b\frac{i \quad 2m \quad 2}{2}c} + C_i^{b\frac{i \quad 2m \quad 1 \quad 2(m+1)}{2}c} & \text{if} \quad i = 2(m+1)+2m+1; \\ C_i^{b\frac{i}{2}c} + C_i^{b\frac{i \quad 2m \quad 1}{2}c} + C_i^{b\frac{i \quad 2m \quad 2}{2}c} + C_i^{b\frac{i \quad 2m \quad 1 \quad 2(m+1)}{2}c} + C_i^{b\frac{i \quad 4(m+1) \quad 2m}{2}c} & \\ & \text{if} \quad 4(m+1) \; 6 \; i \; 6 \; 4(m+1)+2m; \end{cases}$$

$$c_i^m = \begin{cases} 0 & \text{if} \quad 0 \; 6 \; i \; 6 \; m \quad 1; \\ C_i^{b\frac{i \quad m}{2}c} & \text{if} \quad i = m; \\ C_i^{b\frac{i \quad m}{2}c} + C_i^{b\frac{i \quad m \quad 1}{2}c} & \text{if} \quad m+1 \; 6 \; i \; 6 \; m+1+2m; \\ C_i^{b\frac{i \quad m}{2}c} + C_i^{b\frac{i \quad m \quad 1}{2}c} + C_i^{b\frac{i \quad m \quad 1 \quad 2m \quad 1}{2}c} & \text{if} \quad i = m+1+2m+1; \\ C_i^{b\frac{i \quad m}{2}c} + C_i^{b\frac{i \quad m \quad 1}{2}c} + C_i^{b\frac{i \quad m \quad 1 \quad 2m \quad 1}{2}c} + C_i^{b\frac{i \quad 3(m+1)}{2}c} & \text{if} \quad 3(m+1) \; 6 \; i \; 6 \; 3(m+1)+2m; \end{cases}$$

The coefficient in front of $A(k{=}m)$ in (25) takes the following form:

$$\frac{1}{2} \sum_{i=0}^{1} \left(\frac{}{2}\right)^i \left(2^i \quad c_i^0\right) = \frac{1}{2} \sum_{i=0}^{1} \quad {}^i \quad \sum_{i=0}^{1} \frac{{}^i}{2^{i+1}} C_i^{b\frac{i}{2}c} \quad \sum_{i=2m+1}^{1} \frac{{}^i}{2^{i+1}} C_i^{b\frac{i \quad 2m \quad 1}{2}c}$$
$$\sum_{i=2m+2}^{1} \frac{{}^i}{2^{i+1}} C_i^{b\frac{i \quad 2m \quad 2}{2}c} \quad \sum_{i=4m+3}^{1} \frac{{}^i}{2^{i+1}} C_i^{b\frac{i \quad 4m \quad 3}{2}c} \quad ::: = \frac{1}{2(1 \qquad)} \quad \sum_{i=0}^{1} \frac{{}^i}{2^{i+1}} C_i^{b\frac{i}{2}c}$$
$$\sum_{j=0}^{1} \left(\sum_{i=2m+1+2j(m+1)}^{1} \frac{{}^i}{2^{i+1}} C_i^{b\frac{i \quad 2m \quad 1 \quad 2j(m+1)}{2}c} + \sum_{i=2m+2+2j(m+1)}^{1} \frac{{}^i}{2^{i+1}} C_i^{b\frac{i \quad 2m \quad 2 \quad 2j(m+1)}{2}c} \right):$$

We can rewrite the second term:

$$\sum_{i=0}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i}{2}c} = \sum_{i=0}^{1} \frac{2i}{2^{2i+1}} C_{2i}^{i} + \sum_{i=0}^{1} \frac{2i+1}{2^{2i+2}} C_{2i+1}^{i} = \frac{1}{2} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{2i} \left(C_{2i}^{i} + \frac{}{2} C_{2i+1}^{i}\right):$$

The term in parenthesis can also be simplified by considering even and odd indices separately:

$$\sum_{i=2m+1+2j(m+1)}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad 2m \quad 1 \quad 2j(m+1)}{2}c} + \sum_{i=2m+2+2j(m+1)}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad 2m \quad 2 \quad 2j(m+1)}{2}c} =$$

$$= \frac{2(j+1)(m+1) \quad 1}{2^{2(j+1)(m+1)}} + \sum_{i=2(j+1)(m+1)}^{1} \frac{i}{2^{i+1}} \left(C_i^{b\frac{i+1 \quad 2(j+1)(m+1)}{2}c} + C_i^{b\frac{i \quad 2(j+1)(m+1)}{2}c}\right) =$$

$$= \frac{2(j+1)(m+1) \quad 1}{2^{2(j+1)(m+1)}} + \sum_{i=0}^{1} \frac{2(j+1)(m+1)+2i}{2^{2(j+1)(m+1)+2i}} C_{2(j+1)(m+1)+2i}^{i} +$$

$$+ \sum_{i=0}^{1} \frac{2(j+1)(m+1)+2i+1}{2^{2(j+1)(m+1)+2i+2}} C_{2(j+1)(m+1)+2i+2}^{i+1} = \frac{1+}{2} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{2(j+1)(m+1)+2i} C_{2(j+1)(m+1)+2i}^{i}:$$

The coefficient in front of $B(k=m)$ in (25) takes the following form:

$$\frac{1}{2} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{i} (2^{i} \quad c_i^{m}) = \frac{1}{2} \sum_{i=0}^{1} {}^{i} + \sum_{i=m}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad m}{2}c} + \sum_{i=m+1}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad m \quad 1}{2}c}$$

$$+ \sum_{i=3m+2}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad 3m \quad 2}{2}c} + \sum_{i=3m+3}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad 3m \quad 3}{2}c} + :::$$

$$= \frac{1}{2(1 \qquad)} + \sum_{j=0}^{1} \left(\sum_{i=m+2j(m+1)}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad m \quad 2j(m+1)}{2}c} \right.$$

$$\left. + \sum_{i=m+1+2j(m+1)}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad m \quad 2j(m+1) \quad 1}{2}c} \right):$$

The term in parenthesis can be simplified the same way as for $A(k=m)$:

$$\sum_{i=m+2j(m+1)}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad m \quad 2j(m+1)}{2}c} + \sum_{i=m+1+2j(m+1)}^{1} \frac{i}{2^{i+1}} C_i^{b\frac{i \quad m \quad 2j(m+1) \quad 1}{2}c} =$$

$$= \frac{1+}{2} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{(2j+1)(m+1)+2i \quad 1} C_{(2j+1)(m+1)+2i}^{i}:$$

Summing everything up, we obtain the overall expected profit of the storage from (25):

$$U_s^C = \frac{A(k{=}m) \quad B(k{=}m)}{2(1 \quad)} + \frac{1+}{} \left(B\left(\frac{k}{m}\right) \sum_{j=0}^{1} \left(\frac{}{2}\right)^{(m+1)(2j+1)} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{2i} C^i_{2i+(m+1)(2j+1)} \right.$$
$$\left. A\left(\frac{k}{m}\right) \sum_{j=1}^{1} \left(\frac{}{2}\right)^{2(m+1)j} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{2i} C^i_{2i+2(m+1)j} \right) \quad \frac{A(k{=}m)}{2} \sum_{i=0}^{1} \left(\frac{}{2}\right)^{2i} \left(C^i_{2i} + \frac{}{2} C^i_{2i+1} \right):$$

Using formula

$$\sum_{i=0}^{1} {}^i C^i_{2i+r} = \frac{2^r}{\sqrt{1 \quad 4} \left(1+\sqrt{1 \quad 4}\right)^r}$$

from Graham et al. (1994) (p. 203) and introducing new discounting coefficient

$$\tilde{} = \frac{}{1+\sqrt{1 \quad {}^2}};$$

we finally get

$$U_s^C = \frac{1}{2(1 \quad)} \left(\quad B\left(\frac{k}{m}\right) + \tilde{} A\left(\frac{k}{m}\right) \right.$$
$$\left. \frac{2\sqrt{1 \quad {}^2}}{} \frac{\tilde{}^{m+1}}{1 \quad \tilde{}^{2(m+1)}} \left(\quad B\left(\frac{k}{m}\right) + \tilde{}^{m+1} A\left(\frac{k}{m}\right)\right)\right): \tag{26}$$

A.2.3 Proof of Proposition 5

Let's start with the possible deviation of the storage. If the storage decides to enter the market and make a purchase under monopolistic quantities set by generators, it has to sell energy under Cournot quantities. The expected storage payoffs during one round-trip cycle are

$$\left(\frac{1 \quad a}{2} + k\right) k + \frac{}{2} \frac{1+a \quad k}{n+1} \; k:$$

The storage doesn't gain any profits if

$$\frac{}{2} \ 6 \ \frac{(n+1)\left(\frac{1 \quad a}{2}+k\right)}{(1+a \quad k)};$$

which is equivalent to the right side of (12). Hence, as long as generators can maintain their collusive equilibrium, the storage unit should not purchase any energy and so never operates.

Let's analyze the possible deviations of a generator. First, we consider the case when inequality (13) doesn't hold. It means that the storage is not interested in entering the market with Cournot bids. A generator deviates from its monopolistic quantity when the shock is positive: $'' = a$. Let the deviation be . Then the generator's profit after deviating is

$$\left(\frac{1+a}{2n}+ \quad \right)\left(\frac{1+a}{2} \quad \right)+\frac{}{1}\left(\frac{1}{2}\left(\frac{1+a}{n+1}\right)^2+\frac{1}{2}\left(\frac{1 \quad a}{n+1}\right)^2\right):$$

A generator increases its quantity by , which results in the price going down by also. Because of that, all other generators switch from monopolistic quantities to Cournot quantities, which results in the discounted expected payoff over infinite horizon expressed by the second item. This move is unprofitable if

$$\left(\frac{1+a}{2n}+ \quad \right)\left(\frac{1+a}{2} \quad \right)+\frac{}{1}\left(\frac{1}{2}\left(\frac{1+a}{n+1}\right)^2+\frac{1}{2}\left(\frac{1 \quad a}{n+1}\right)^2\right) \ 6$$
$$6 \ \frac{1+a}{2n}\frac{1+a}{2}+\frac{}{1}\left(\frac{1}{2}\frac{(1+a)^2}{4n}+\frac{1}{2}\frac{(1 \quad a)^2}{4n}\right);$$

which can be simplified to

$$\frac{n \quad 1}{n}(1+a) \qquad 2 \ ^2 \ 6 \ \frac{}{1}\frac{(1+a^2)(n \quad 1)^2}{2n(n+1)^2}:$$

The maximum on the left side can be achieved when $_{max} = (1+a)(n \quad 1)=(4n)$. Then we obtain

$$\frac{(1+a)^2}{4n} \ 6 \ \frac{}{1}\frac{(1+a^2)}{(n+1)^2}; \tag{27}$$

which is equivalent to the left side of (12).

Now assume that a generator deviates from its monopolistic quantity when the shock is negative: $'' = \quad a$. This move is unprofitable if

$$\left(\frac{1 \quad a}{2n} + \quad \right)\left(\frac{1 \quad a}{2} \quad \right) + \frac{}{1 \quad}\left(\frac{1}{2}\left(\frac{1+a}{n+1}\right)^2 + \frac{1}{2}\left(\frac{1 \quad a}{n+1}\right)^2\right) 6$$

$$6 \ \frac{1 \quad a}{2n}\frac{1 \quad a}{2} + \frac{}{1 \quad}\left(\frac{1}{2}\frac{(1+a)^2}{4n} + \frac{1}{2}\frac{(1 \quad a)^2}{4n}\right);$$

which can be simplified to

$$\frac{(1 \quad a)^2}{4n} \ 6 \ \frac{}{1 \quad}\frac{1+a^2}{(n+1)^2}:$$

This inequality is weaker than (27).

Now consider the case when inequality (13) holds. It means that the storage finds profitable to enter the market immediately after any of the generators has deviated. Since the payoffs of a deviating generator with the participating storage are lower than the ones without it, the inequality (27) is sufficient to make this deviation unprofitable.

Finally, we should also prove that our pool of equilibrium strategies forms SPNE even on off the equilibrium path information sets. Namely, generators must not want to deviate even if the storage enters the market. Indeed, if the shock is positive, all the generators set static Cournot quantities, and it becomes unprofitable to deviate. If the shock is negative, any deviation from monopolistic quantities implies punishment that was already considered earlier. Hence, all the possible deviations of a generator are unprofitable, and the proposed strategies of all the players form Nash equilibrium.

To calculate the expected payoff of a generator, we obtain a recursive equation for generator's payoff Z_t starting from period t:

$$Z_t = \frac{1}{2}G_{01} + \frac{1}{2}G_{10} + \quad Z_{t+1}:$$

We can easily find $Z_0 = U_g^0$ from this equation.

Since positive and negative shocks are equally likely, the cumulative consumers' expected payments per period are

$$C^0 = \frac{1}{2}\left(\frac{1+a}{2}\right)^2 + \frac{1}{2}\left(\frac{1-a}{2}\right)^2 = \frac{1+a^2}{4}.$$

A.2.4 Proof of Proposition 6

To justify inequalities (15), (16), and (17), we need to find the expected payoffs of the storage unit. Let the value function $V_t^{\{-,+\}}(i)$, $i \in \{0, k/2, k\}$ be the total expected payoff of the storage operator from t on if the current state is empty ($i = 0$), half-full ($i = k/2$), or full ($i = k$) and the current shock is either negative ($-$) or positive ($+$). We have a system of recursive equations:

$$\begin{cases} V_t^-(k) = y \; V_{t+1}^-(k) + (1-y)\left(A\left(\frac{k}{2}\right) + V_{t+1}^+\left(\frac{k}{2}\right)\right); \\ V_t^-\left(\frac{k}{2}\right) = y\left(-B\left(\frac{k}{2}\right) + V_{t+1}^-(k)\right) + (1-y)\left(A\left(\frac{k}{2}\right) + V_{t+1}^+(0)\right); \\ V_t^+\left(\frac{k}{2}\right) = (1-x)\left(-B\left(\frac{k}{2}\right) + V_{t+1}^-(k)\right) + x\left(A\left(\frac{k}{2}\right) + V_{t+1}^+(0)\right); \\ V_t^+(0) = (1-x)\left(-B + V_{t+1}^-\left(\frac{k}{2}\right)\right) + x \; V_{t+1}^+(0). \end{cases}$$

for any integer $t > 0$. It can be rewritten in a matrix form

$$V_t = P_2 + Q_2 \; V_{t+1}, \tag{28}$$

where

$$V_t = \begin{pmatrix} V_t^-(k) \\ V_t^-(k/2) \\ V_t^+(k/2) \\ V_t^+(0) \end{pmatrix}; \quad P_2 = \begin{pmatrix} (1-y)A(k/2) \\ (1-y)A(k/2) - yB(k/2) \\ xA(k/2) - (1-x)B(k/2) \\ -(1-x)B(k/2) \end{pmatrix}; \quad Q_2 = \begin{pmatrix} y & 0 & 1-y & 0 \\ y & 0 & 0 & 1-y \\ 1-x & 0 & 0 & x \\ 0 & 1-x & 0 & x \end{pmatrix}.$$

To calculate power t of matrix Q_2, we find the Jordan decomposition $Q_2 = T \; J \; T^{-1}$ of

Q_2. Here,

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{(1\quad x)(1\quad y)} & 0 & 0 \\ 0 & 0 & \sqrt{(1\quad x)(1\quad y)} & 0 \\ 0 & 0 & 0 & x+y\quad 1 \end{pmatrix};$$

$$T = \begin{pmatrix} 1 & \frac{\sqrt{\frac{\frac{x\sqrt{1\ y}}{y\sqrt{1\ x}}}{(1\ x)(1\ y)\ \ x}}}{1\ x} & \frac{\sqrt{\frac{\frac{x\sqrt{1\ y}}{y\sqrt{1\ x}}}{(1\ x)(1\ y)+x}}}{1\ x} & \frac{1\ y}{1\ x} \\ 1 & & & \frac{1\ y}{1\ x} \\ 1 & \sqrt{\frac{x}{(1\ x)(1\ y)}}\quad \frac{x}{y} & \sqrt{\frac{x}{(1\ x)(1\ y)}}\quad \frac{x}{y} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix};$$

so $Q_2^t = T \quad J^t \quad T \quad ^1$.

For $\quad < 1$, we can find from (28) that

$$V_0 = \sum_{i=0}^{t} \quad {}^iQ_2^i \quad P_2 + \quad {}^{t+1}Q_2^{t+1} \quad V_{t+1} \quad \underset{t\ !\ 1}{!} \quad \sum_{i=0}^{1} \quad {}^iQ_2^i \quad P_2 =$$

$$= T \begin{pmatrix} \frac{1}{1\quad} & 0 & 0 & 0 \\ 0 & \frac{1}{1\quad \sqrt{\frac{1}{(1\ x)(1\ y)}}} & 0 & 0 \\ 0 & 0 & \frac{1}{1+\quad \sqrt{\frac{1}{(1\ x)(1\ y)}}} & 0 \\ 0 & 0 & 0 & \frac{1}{1+\ \ (1\ \ x\ \ y)} \end{pmatrix} T \quad ^1 \quad P_2 =$$

$$= \frac{1}{(1\quad)(1\quad d)\,(1\quad {}^2(1\quad x)(1\quad y))}$$

$$\begin{pmatrix} (1\quad y)\left(\quad (1\quad x)(1+\quad {}^2d)B\left(\frac{k}{2}\right) + \left(1\quad (1\quad y+xd)\quad {}^2\right)A\left(\frac{k}{2}\right)\right) \\ \left((1\quad y)(y\quad d) + \quad {}^3(1\quad x)(1\quad y)d\right)B\left(\frac{k}{2}\right) + (1\quad y)(1\quad x)(1+\quad {}^2d)A\left(\frac{k}{2}\right) \\ (1\quad x)(1\quad y)(1+\quad {}^2d)B\left(\frac{k}{2}\right) + \left((1\quad x)(x\quad d) + \quad {}^3(1\quad x)(1\quad y)d\right)A\left(\frac{k}{2}\right) \\ (1\quad x)\left(\quad \left(1\quad (1\quad x+yd)\quad {}^2\right)B\left(\frac{k}{2}\right) + \quad (1\quad y)(1+\quad {}^2d)A\left(\frac{k}{2}\right)\right) \end{pmatrix}:$$

from where we finally get $U_s^2 = V_0^+(0)$:

$$U_s^2 = \frac{1 \quad x}{(1 \quad)(1 \quad d)} \left(\quad B\left(\frac{k}{2}\right) + \quad (1 \quad y)A\left(\frac{k}{2}\right) + \quad {}^2 y \, \frac{yB\left(\frac{k}{2}\right) + \quad x(1 \quad y)A\left(\frac{k}{2}\right)}{1 \quad {}^2(1 \quad x)(1 \quad y)} \right):$$

Storage operates in this market only if $U_s^2 > 0$, which is exactly inequality (16).

For Condition 12, equation (28) reads

$$V_t = P_3 + \quad Q_3 \quad V_{t+1};$$

where

$$V_t = \begin{pmatrix} V_t \ (k) \\ V_t \ (2k{=}3) \\ V_t^+(2k{=}3) \\ V_t \ (k{=}3) \\ V_t^+(k{=}3) \\ V_t^+(0) \end{pmatrix}; \qquad P_3 = \begin{pmatrix} (1 \quad y)A(k{=}2) \\ (1 \quad y)A(k{=}2) \quad yB(k{=}2) \\ (1 \quad y)A(k{=}2) \quad yB(k{=}2) \\ xA(k{=}2) \quad (1 \quad x)B(k{=}2) \\ xA(k{=}2) \quad (1 \quad x)B(k{=}2) \\ (1 \quad x)B(k{=}2) \end{pmatrix};$$

$$Q_3 = \begin{pmatrix} y & 0 & 0 & 1 \quad y & 0 & 0 \\ y & 0 & 0 & 0 & 1 \quad y & 0 \\ 0 & y & 0 & 0 & 0 & 1 \quad y \\ 1 \quad x & 0 & 0 & 0 & x & 0 \\ 0 & 1 \quad x & 0 & 0 & 0 & x \\ 0 & 0 & 1 \quad x & 0 & 0 & x \end{pmatrix}:$$

The six eigenvalues of matrix Q_3 that compose the diagonal of the corresponding Jordan matrix J are

$$_1 = 1; \qquad _2 = x + y \quad 1; \qquad _{3;4;5;6} = \quad \sqrt{(1 \quad x)(1 \quad y) \quad \sqrt{xy(1 \quad x)(1 \quad y)}}:$$

Following the same argumentation as in case of k=2, we get

$$V_0^+(0) = \frac{1 \quad x}{(1 \qquad)(1 \qquad d)} \left[\quad B\left(\frac{k}{3}\right) + \quad (1 \quad y)A\left(\frac{k}{3}\right) \right.$$

$$\left. + \ ^3 y \ \frac{y^2 B\left(\frac{k}{3}\right) + x(1 \quad y)(1 + \ ^2 d)A\left(\frac{k}{3}\right)}{(1 \qquad ^2(1 \quad x)(1 \quad y))^2 \qquad ^4 xy(1 \quad x)(1 \quad y)} \right]:$$

Inequality $V_0^+(0) = U_s^3 > 0$ is exactly (17).

Four possible deviations of the storage should be considered. All other deviations are just compositions of those four.

- A storage unit that is not empty deviates by not selling under the positive shock. Then there may be only loss comparing to the default strategy. Indeed, nothing changes on the market except the future profits to be discounted by .

- A storage unit that is not full deviates by not buying under the negative shock. Also, no gains here.

- A full storage unit deviates by selling under the negative shock. In this situation, the quantities supplied by the generators are $q = (1 \quad a)=(n + 1)$. The resulting price after the deviation are

$$p = 1 \quad a \qquad \frac{k}{m} \quad n\frac{1 \quad a}{n+1} = \frac{1 \quad a}{n+1} \qquad \frac{k}{m}:$$

  To make this deviation profitable, the storage operator must gain more than if it waits for the positive shock and sells in that period, but this contradicts (16) or (17).

- The nonfull storage unit deviates by buying under the positive shock. Here we have $q = (1+a)=(n+1)$, and the resulting price after the deviation is $p = (1+a)=(n+1)+k=m$. It is easy to verify that the corresponding payoff is strictly negative.

Next we must rule out possible deviations of the generators. In each round, we have a static Cournot equilibrium for all the participants. Any change of the equilibrium quantity in round t leads to decreasing the payoffs in that round and, thus, decreasing the overall payoffs.

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