Modeling trading over time

Sergei Balakin[*] & Guillaume Roger[*]

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Abstract

An analyst can select among many instruments to model an economic phenomenon. For a dynamic setting, we compare the performance of a model that relies on quantities (real numbers) to one built on flows (functions of time). Naturally flows account better for the timing of trade and variations in demand. Moreover, flows allow the trader to define what is a buying or selling interval independently of calendar time, which is ruled out by construction if relying on quantities. This has implications for the quantities traded at the optimum; flows induce as much as 8% more trade in our parametrization.

Keywords: *dynamic trading, storage, market power*

JEL: *C73, D43, D47, Q41, Q42*

1 Introduction

A model is an abstract simplification of reality and so must necessarily leave out some details. Which of these simplifications to adopt is part science and part art. Sometimes these choices can be validated: for example, Bushnell et al. (2008) show that the Cournot model (of quantities) is an empirically acceptable approximation of the richer, but too cumbersome, model of supply functions (see Klemperer and Meyer (1989), among others).

In this paper we ask a similar question: when modeling a dynamic problem, what are the consequences of restricting attention to quantities (real numbers) that typically arise in

[*]Monash University. Funding from ARENA under grant 2021/ARP-016 gratefully acknowledged. Corresponding author: Roger–guillaume.roger72@gmail.com

a discrete model rather than adopting *flows*–continuous functions of time that integrate to quantities? Our motivating example is that of an electricity market: power flows are injected and withdrawn continuously over time and integrate to a quantity of energy. It may be natural then to explicitly model these power flows at the cost of handling a functional problem. But if thinking of trading electricity over days or weeks, can the analyst rely on a simpler model of quantities traded over discrete periods and sidestep the complexity of functional analysis? In other words, how much, if anything, is lost by using a simpler model?

As often in economics, it depends. We model the merchant trading of a commodity (for example, electricity) over time. During a low-period demand, a storage unit buys the commodity to then sell it when demand is high. How high is the demand is subject to an uncertain shock ε. Based on her information, the operator computes how much to sell, given that she must also first buy that commodity. Our results do not hinge on this being a model of storage, but it has the benefit of allowing for more cases of interest. What matters for our result is the interaction of the design of the optimal trading strategy with the timing of resolution of this uncertainty. How much is lost from using the simpler model depends on the information the players have access to and on the skewness of realized demand.[1]

When players only ever know the distribution of uncertainty, trading in continuous flows or discrete quantities delivers the same allocations. The intuitive reason is that players can only make decisions based on averages: average quantities when buying and selling, average flows also when buying and selling, and, with flows, a *single* selling window based on averages. If the quantity trader could first commit to her optimal quantities and then use the same functions as the flow trader for each sub-period, there would be no difference at all including prices.

This result changes if players have access to information *after* buying but *before* selling. Then, upon selling, the trader can use the strategy that is most appropriate to the demand realization (and the quantity available to sell). With flows, this is a function $s(t, \varepsilon)$ that follows the path of demand given ε, including *when* exactly to trade: there is a continuum of trading windows–one for each realization of ε. With discrete quantities $S(\varepsilon)$, the notion of *when* to trade (during that period) is irrelevant. This distinction matters: when computing

[1]We establish this claim using a parametrized formulation of demand and so acknowledge it may also depend on the exact definition of demand.

the expected traded quantity over the support of ε, we find that the monopolist trades more using flows. Using continuous flows delivers a better allocation thanks to this nuance: the precise timing of trade does matter. Here it introduces some convexity that does not just vanish through integration (of uncertainty). Returning to our comparison, if the quantity trader were to commit to her optimal quantities and then use the optimal buying and selling functions, she would trade as much as under no information– but less than the flow trader. She would also generate less surplus.

This second result becomes even starker if players have access to (all) information *before* buying–the uncertainty is removed. The trader need no longer average over the support of the uncertainty ε when buying. She enjoys the full benefit of exactly adjusting the timing of sale and of purchase to the demand conditions. Moreover, the more dispersed are the demand shocks, the more acute is the difference between trading in quantities and trading in flows.

We also explore an important, complementary question that can only be asked of a continuous-time model: what exactly determines a "buying period" and a "selling period"? Using quantities implies relying on a discrete time model, where the grid (exogenously) determines buying and selling periods. With continuous flows, the trader endogenously decides when to buy and sell. A low demand shock (whether anticipated or realized) in the high-demand period induces the trader to stop buying, and start selling sooner, to mitigate her own market power. Conversely, a high demand shock in the high-demand period induces her to delay selling to extract higher prices, and continue buying instead. The converse holds for shocks to the low-demand period. Quite naturally, the timing and duration of buying and selling periods are not determined by calendar time but by demand conditions over time.

As ever, which approach to adopt depends on the problem on hand, including the information structure. For example, if studying a long-term problem, the minute details of intra-period trading may not be the most relevant. Then using the simpler quantities model may be appropriate. In contrast, if results are sensitive to the presence of constraints that may bind at times, using flows is better suited as both more accurate and time-responsive.

This paper is closest to a companion paper Balakin and Roger (2025), in which we study a dynamic problem of electricity trading in continuous time using power flows as actions. In that paper we are interested in intertemporal linkages between periods. The present work also bears connection to Andres-Cerezo and Fabra (2025), who also use power flows and a periodic function to model demand. How to trade over time is a question of importance in finance, starting with the seminal model of Kyle (1985), who was mostly interested in the informational content of prices. Vayanos (1999) adds price impact to Kyle (1985) and Almgren and Chriss (2001) introduce a trade-off between price impact and uncertainty. Sannikov and Skrzypacz (2023) extend these models to asymmetric traders. There are many more; still, in all these papers, the tension stems from the information that trading generates. We offer a complementary perspective: when to trade matters, irrespective of informational consequences.

Next we introduce the models we compare. In the following section we carry out the analysis in a nested fashion. Last we conclude. All proofs are relegated to the Appendix.

2 Models of dynamic trading

Consider a market with a single storage operator, an infinite number of competitive sellers (for example, generators in the case of electricity), and a pool of consumers. Throughout, the storage unit has a large, but finite inventory capacity $k < \infty$. Whenever the capacity k is the constraining factor, at most k is traded whether using flows or relying on quantities, and this paper has no object. We focus on the alternative case. As a consequence, we ignore the capacity constraint throughout this paper, including in our proofs. We also ignore efficiency losses, which are orthogonal to the argument we wish to make.

The sellers each produce q with a diminishing-return technology that induces convex costs. Specializing the cost function to be quadratic is sufficient for the model and not substantive to the argument, so $c(q) := q^2/2$. By the standard arguments that $Marginal\ Cost = Marginal\ Benefit$ and $MB = p$ due to perfect competition among suppliers, the clearing price p is equal to the aggregate quantity $q = c'(q)$.

The object to this paper is to explore whether trading using flows $b(t), s(t)$ that are functions of time, rather than simpler quantities B, S, makes a difference. To do so we have to consider two specifications; one that uses flows and the other quantities. These models are made to coincide in that the time-integral of power flows is a quantity.

2.1 Time-continuous trading

The behavior of consumers is described by the price inelastic, but time-varying, demand function $D(t, \varepsilon_1, \varepsilon_2)$, also parametrized by θ such that, for each cycle,[2]

$$D(t, \theta, \varepsilon_1, \varepsilon_2) = \begin{cases} \theta - (1 + \varepsilon_1) \sin t & \text{if } \ t \in [0, \pi), \\ \theta - (1 + \varepsilon_2) \sin t & \text{if } \ t \in [\pi, 2\pi), \end{cases} \tag{1}$$

and which is depicted in Figure 1. The term $\theta \geqslant 1$ is a parameter that captures the mean demand, t is time, and $\varepsilon_1 \sim \mathrm{U}_{[-l_1, l_1]}, \varepsilon_2 \sim \mathrm{U}_{[-l_2, l_2]}$ are demand shocks that augment the trough and peak of demand, respectively.

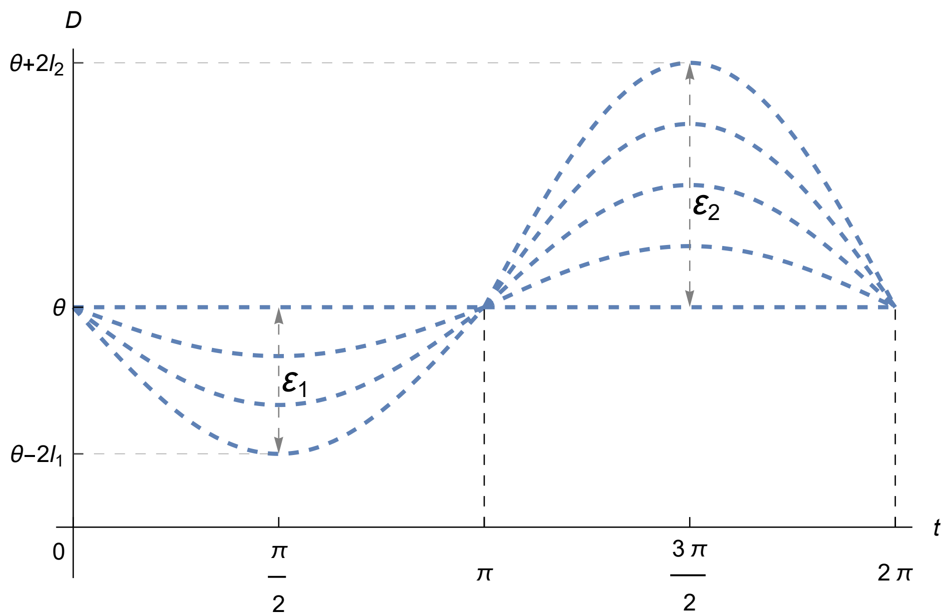

Figure 1: Demand function for $0 \leqslant t \leqslant 2\pi$.

Although in general the demand function D is periodic, we focus on a single period, which is sufficient to our argument.[3] Therefore the shock ε_1 is commonly known and acts as a parameter of this model. In consequence, for most of the paper, we can normalize $\varepsilon_1 = 0$ and $l_2 = 1$. Let also ε_2 be denoted simply as ε where it does not lead to confusion. The demand (1) simplifies accordingly. We consider other values for ε_1 and l_2 in Section 3.4.

[2]It largely reflects the empirical evidence in most power markets that exhibit this systematic demand pattern (called the "duck curve"), such as California, Texas, Spain or Australia. The structure of this uncertainty can also apply to many commodities where supply conditions are known but (future) demand uncertain.

[3]The shocks ε can be interpreted either strictly as a demand shock or as a supply shock, with $D(t, \varepsilon_1, \varepsilon_2)$ being net demand.

In this specification, the storage unit can only buy *flows* $b(t)$ or sell $s(t)$. Given $D(t,\varepsilon)$, one can reasonably conjecture that the storage unit buys $b(t),\ t \in [0,\pi)$ under low demand conditions and sells $s(t),\ t \in [\pi, 2\pi)$ when demand is high. This conjecture is actually a substantive assumption, which we also relax in Section 3.4. Relying on it at this point allows us to break the argument and ease the exposition.

Combining the behaviors of sellers and of storage, we obtain the following market clearing conditions that determine the price function p:

$$q(t) = p(t) = \theta - \sin t + b(t) \text{ if } t \in [0,\pi), \tag{2}$$

$$q(t,\varepsilon) = p(t,\varepsilon) = \theta - (1+\varepsilon)\sin t - s(t) \text{ if } t \in [\pi, 2\pi). \tag{3}$$

These equations make it plain that a storage operator seeks to arbitrage price differences and that the inverse demand faced by the storage unit is elastic.

2.2 Discrete trading: quantities

In the second (simplified) specification, the operator first buys the quantity B at the demand level D_1 and then sells the quantity S at the demand level D_2. Periods 1 and 2 correspond to the first phase (0 to π) of the cycle, and the second one (π to 2π), respectively. To make these two cases fully comparable, we perform some normalization with respect to the demand (1):

$$D_1 = \int_0^{\pi} (\theta - \sin t)\, dt = \pi\theta - 2, \qquad D_2 = \int_{\pi}^{2\pi} (\theta - (1+\varepsilon)\sin t)\, dt = \pi\theta + 2(1+\varepsilon),$$

still with $l_1 = 0, l_2 = 1$. In this simpler case, market clearing becomes

$$q_1 = p_1 = \pi\theta - 2 + B, \tag{4}$$

$$q_2(\varepsilon) = p_2(\varepsilon) = \pi\theta + 2(1+\varepsilon) - S. \tag{5}$$

We compare optimal strategies under either specification across different information structures. The reader should also bear in mind that a storage unit cares only about the *difference* in demand and not the level of demand. This is perhaps most apparent from (4) and (5): $p_2(\varepsilon) - p_1 = 4 + 2\varepsilon - (S + B)$.

3 Analysis

The analysis runs sequentially in three parts starting with "no information": the storage unit never learns the realization of ε. This is used as a benchmark. It then continues with "incomplete" information, whereby the shock is discovered between the two trading periods only. Finally, under "full information", the shock is known at time $t = 0$.

3.1 No information

Under no information about the shock ε, the monopolistic storage can only rely on the expected value of the (single) shock ε.

If using quantities, the payoff function reads

$$\max_{B,S} \left[- (\pi\theta - 2 + B) B + \frac{1}{2} \int_{-1}^{1} (\pi\theta + 2(1+\varepsilon) - S) S d\varepsilon \right]. \tag{6}$$

and reflects the fact that there is no uncertainty in the first period but there is in the second one. The support of the random variable ε is $[-1, 1]$ and the density is $1/2$. The storage unit is subject to the natural constraints:

$$0 \leqslant B, \qquad 0 \leqslant S, \qquad S \leqslant B. \tag{7}$$

There is no short-selling and the storage operator cannot sell more than has been bought before. A storage unit starts from empty at "time 0"–the beginning of period 1. Integrating the noise ε, we may simplify the expression (6) to

$$\max_{B,S} \left[- (\pi\theta - 2 + B) B + (\pi\theta + 2 - S) S \right].$$

It is easy to see that to maximize payoff, storage must sell as much as it buys, and that the solution to (6) subject to (7) is $S_n = B_n = 1$. Immediately we may also conclude that if the capacity k is lower than 1, the storage unit trades exactly k. If the capacity exceeds 1, storage still only trades 1. This is the unconstrained monopoly quantity in this model.

When using flows $b(t), s(t)$, the payoff function writes:

$$\max_{b(t),s(t)} \left[- \int_{0}^{\pi} (\theta - \sin t + b(t)) b(t) dt + \frac{1}{2} \int_{-1}^{1} \int_{\pi}^{2\pi} (\theta - (1+\varepsilon) \sin t - s(t)) s(t) dt d\varepsilon \right] \tag{8}$$

now subject to the constraints

$$b(t) \geqslant 0, \qquad s(t) \geqslant 0, \qquad \int_0^{2\pi} (b(t) - s(t))\, dt \geqslant 0, \tag{9}$$

that have the same interpretation as before. Now the objective function (8) can be simplified to

$$\max_{b(t),s(t)} \left[-\int_0^{\pi} (\theta - \sin t + b(t))\, b(t)dt + \int_{\pi}^{2\pi} (\theta - \sin t - s(t))\, s(t)dt \right],$$

that is, with symmetric noise around the mean, the storage unit sells following mean demand. In the Appendix we prove

Lemma 1. *The optimal storage strategy that solves (8) subject to (9) is characterized by*

$$b(t) = \begin{cases} \frac{\sin t}{2} & \textit{if} \;\; t \in [0, \pi), \\ 0 & \textit{otherwise,} \end{cases} \qquad s(t) = \begin{cases} -\frac{\sin t}{2} & \textit{if} \;\; t \in [\pi, 2\pi), \\ 0 & \textit{otherwise.} \end{cases}$$

The storage operator trades the monopoly quantity of 1.

With flows, the storage unit computes the best trajectories $b(t)$ and $s(t)$ to maximize its expected payoff from 0 to 2π; that payoff is max $\mathbb{E}[profit(\varepsilon)]$. It depends only on the *difference* between the low and high demand. The trader simply behaves like a standard monopolist, with the caveat that the optimal strategy spreads the quantities she buys and sells over the relevant interval. We provide more commentary in our companion paper Balakin and Roger (2025).

Even though introducing flows makes for a richer problem, the important result to highlight here is that the traded quantities are the same across both specifications. Hence a simplification to quantities is entirely innocuous under this information structure.

3.2 Incomplete information

Now let the storage operator receive a fully informative signal about the shock ε at $t = \pi$; this could be observing the shock itself or receiving a precise report of it. Clearly the storage unit can set its selling strategy according to this information.

Trading in quantities. Since operating under full information when selling, that quantity may now depend on ε: $S = S(\varepsilon)$. The payoff function writes

$$\max_{B,\{S(\varepsilon)\}} \left[-(\pi\theta - 2 + B)\, B + \frac{1}{2}\int_{-1}^{1} (\pi\theta + 2(1+\varepsilon) - S(\varepsilon))\, S(\varepsilon) d\varepsilon \right], \tag{10}$$

subject to

$$0 \leqslant B, \qquad\qquad 0 \leqslant S(\varepsilon) \leqslant B \qquad \text{for any } \varepsilon \in [-1,1]. \tag{11}$$

In the objective function (10), each selling quantity $S(\varepsilon)$ is a function of the noise ε. Hence we seek a quantity B and a family of functions $\{S(\varepsilon)\}_{-1\leqslant\varepsilon\leqslant 1}$ that solve Problem (10) subject to the constraint set (11). Then we can show

Lemma 2. *The optimal storage strategy that solves* (10) *subject to* (11) *is characterized as*

$$B = S(\varepsilon) = 1 \qquad \textit{for any} \quad \varepsilon \in [-1,1].$$

At most a quantity of 1 is traded; acquiring information has no effect. The reason is that the storage operator must first buy the commodity under incomplete information, and therefore only purchases the average quantity. Buying is the constraining action.

If using flows, the selling strategy becomes $s(t) := s(t,\varepsilon)$. Then the maximization problem changes to:

$$\max_{b(t),\{s(t,\varepsilon)\}} \left[-\int_0^{\pi} (\theta - \sin t + b(t))\, b(t) dt + \frac{1}{2}\int_{-1}^{1}\int_{\pi}^{2\pi} (\theta - (1+\varepsilon)\sin t - s(t,\varepsilon))\, s(t,\varepsilon) dt d\varepsilon \right], \tag{12}$$

subject to constraints

$$b(t) \geqslant 0, \qquad s(t,\varepsilon) \geqslant 0, \qquad \int_0^{2\pi} (b(t) - s(t,\varepsilon))\, dt \geqslant 0 \quad \text{for any } \varepsilon \in [-1,1]. \tag{13}$$

Now, in the objective function (12), each trajectory is a function $s(t,\varepsilon)$ of the noise ε. Again, this makes for a much richer problem. Not only can the operator use a selling strategy that is perfectly adapted to her information, we show she can even trade *more* energy.

Lemma 3. *The optimal storage strategy that solves* (12) *subject to* (13) *is characterized as:*

$$b(t) = \begin{cases} \frac{\sin t}{2} + \frac{1}{\pi}(c_m - 1) & \textit{if } \ t \in [0, \pi], \\ 0 & \textit{otherwise}, \end{cases}$$

$$s(t,\varepsilon) = \begin{cases} -\frac{1+\varepsilon}{2}\left(\sin t + \sin t_\varepsilon\right) & \textit{if } \ t \in [\pi + t_\varepsilon, 2\pi - t_\varepsilon), \\ 0 & \textit{otherwise}, \end{cases} \qquad \textit{if} \qquad \varepsilon \geqslant c_m - 1,$$

$$s(t,\varepsilon) = \begin{cases} \frac{1}{\pi}\left(c_m - 1 - \varepsilon\right) - \frac{1}{2}\left(1 + \varepsilon\right)\sin t & \textit{if } \ t \in [\pi, 2\pi), \\ 0 & \textit{otherwise}, \end{cases} \qquad \textit{if} \qquad \varepsilon < c_m - 1,$$

and where the real number $c_m \approx 1.009$ *and the* function $t_\varepsilon \in [0, \pi/2]$ *are jointly determined as roots of the system of equations*

$$\begin{cases} \displaystyle\int_{c_m - 1}^{1} (1+\varepsilon)\sin t_\varepsilon d\varepsilon = \frac{4}{\pi}(c_m - 1) + \frac{c_m^2}{\pi}, \\ \cos t_\varepsilon - \left(\dfrac{\pi}{2} - t_\varepsilon\right)\sin t_\varepsilon = \dfrac{c_m}{1+\varepsilon}. \end{cases} \tag{14}$$

In our companion paper Balakin and Roger (2025) we offer an extensive commentary of this characterization. Here we merely point out, first, that c_m exceeds 1: the average quantity traded using flows exceeds that using quantities. The difference is not large with this parametrization – approximately 1% – but it exists, and we expand on this point when considering asymmetric shock realizations $\varepsilon_1, \varepsilon_2$ in Section 3.4. Second, the function $s(t,\varepsilon)$ differs from $s(t)$ (characterized in Lemma 1): it explicitly depends on the term ε and the threshold t_ε. Figure 2 provides some intuition for the larger quantities: no matter the exact realization, under flows the trader can always find a function $s(t,\varepsilon)$ that follows the path of demand, including when exactly to sell as determined by t_ε. This flexibility induces some convexity in the expected revenue, so that the trader seeks to buy more. Here too, buying is the constraining action: buying more increases the marginal cost.

Figure 2 shows a family of functions $s(t,\varepsilon)$ for different realizations of ε. For $\varepsilon > c_m - 1$, the support of $s(t,\varepsilon)$ lies strictly inside the interval $[\pi, 2\pi]$. This support expands (and the peak of the curve decreases) as ε decreases. At $\varepsilon = c_m - 1$, the support reaches its natural boundaries and can no longer expand. This is the first line for $s(t,\varepsilon)$ in Lemma 3. As ε keeps decreasing, the linear component of $s(t,\varepsilon)$ kicks in–this is the second line. The "wings" elevate with a discrete jump at each boundary until the curve becomes the horizontal line c_m/π for

$\varepsilon = -1$. We comment extensively and provide more intuition on the behavior of optimal strategies $b(t)$ and $s(t, \varepsilon)$ in Balakin and Roger (2025).

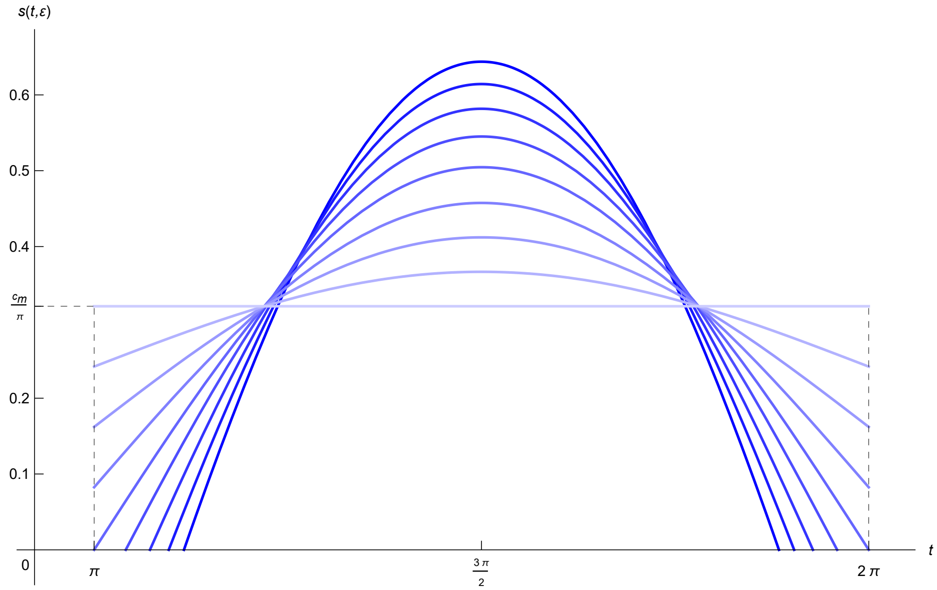

Figure 2: The optimal selling strategies $s(t, \varepsilon)$ for different ε with incomplete information.

3.3 Full information

Suppose now that the storage operator knows the realization of the second shock ε from the very beginning of the cycle–at $t = 0$.[4] This analysis, in spite of its simplicity, delivers a surprising result.

The quantities now all depend on the information ε because the storage operator knows the shock at time $t = 0$, so $S := S(\varepsilon)$ and $B := B(\varepsilon)$. The payoff function writes

$$\max_{B(\varepsilon), S(\varepsilon)} \left[-\left(\pi\theta - 2 + B(\varepsilon)\right) B(\varepsilon) + \left(\pi\theta + 2(1+\varepsilon) - S(\varepsilon)\right) S(\varepsilon) \right], \tag{15}$$

subject to the modified constraint set

$$0 \leqslant B(\varepsilon), \qquad 0 \leqslant S(\varepsilon) \leqslant B(\varepsilon) \qquad \text{for any } \varepsilon \in [-1, 1]. \tag{16}$$

The shock ε can now be treated as a known exogenous parameter of the model. The next lemma holds.

Lemma 4. *The optimal storage strategy that solves (15) subject to (16) is characterized as*

$$B(\varepsilon) = S(\varepsilon) = 1 + \frac{\varepsilon}{2}. \tag{17}$$

[4] We continue with the simplified notation for $\varepsilon_2 \equiv \varepsilon$.

Clearly, depending on ε, the traded quantity changes linearly from $1/2$ (the lowest shock) to $3/2$ (the highest shock). However the expected value remains equal to 1.

When trading in flows, the buying and selling functions both depend on ε: $b := b(t,\varepsilon)$ and $s := s(t,\varepsilon)$. The storage operator now solves:

$$\max_{b(t,\varepsilon),s(t,\varepsilon)} \left[-\int_0^{\pi} (\theta - \sin t + b(t,\varepsilon))\, b(t,\varepsilon) dt + \int_{\pi}^{2\pi} (\theta - (1+\varepsilon)\sin t - s(t,\varepsilon))\, s(t,\varepsilon) dt \right], \tag{18}$$

subject to the now familiar constraints

$$0 \leqslant b(t,\varepsilon), \qquad 0 \leqslant s(t,\varepsilon), \qquad \int_0^{2\pi} (b(t,\varepsilon) - s(t,\varepsilon))\, dt \geqslant 0 \qquad \text{for any } \varepsilon \in [-1,1]. \tag{19}$$

With this we can show:

Lemma 5. *The optimal storage strategy that solves (18) subject to (19) is characterized as:*

- *for* $-1 \leqslant \varepsilon < 0$,

$$b(t,\varepsilon) = \begin{cases} \frac{1}{2}(\sin t - \sin t_b) & \text{if } t \in [t_b, \pi - t_b], \\ 0 & \text{otherwise}, \end{cases}$$

$$s(t,\varepsilon) = \begin{cases} \frac{1}{2}(-(1+\varepsilon)\sin t + \sin t_b) & \text{if } t \in [\pi, 2\pi], \\ 0 & \text{otherwise}; \end{cases}$$

- *for* $0 \leqslant \varepsilon \leqslant 1$,

$$b(t,\varepsilon) = \begin{cases} \frac{1}{2}(\sin t + (1+\varepsilon)\sin t_s) & \text{if } t \in [0, \pi], \\ 0 & \text{otherwise}, \end{cases}$$

$$s(t,\varepsilon) = \begin{cases} -\frac{1+\varepsilon}{2}(\sin t + \sin t_s) & \text{if } t \in [\pi + t_s, 2\pi - t_s], \\ 0 & \text{otherwise}; \end{cases}$$

where $t_b, t_s \in [0, \pi/2]$ *are defined as roots, respectively, of the equations*

$$\cos t_b - (\pi - t_b)\sin t_b = 1 + \varepsilon, \tag{20}$$

$$\cos t_s - (\pi - t_s)\sin t_s = \frac{1}{1+\varepsilon}. \tag{21}$$

We can see that for negative shocks, the operator tends to buy less (and thus to start later at $t_b > 0$) and to start selling everything earlier because the demand peak is not that high. The converse holds for positive shocks: the trader starts buying a lot immediately and waits to sell it later at $\pi + t_s$ ($t_s > 0$) to extract higher profits under the high demand peak.

Here too we can contrast the volumes traded under flows and quantities. Not surprisingly, for $\varepsilon = 0$, the demand levels are symmetric and the traded quantities are always 1. This is exactly the monopoly quantity. However these quantities change as demand conditions become more extreme, as shown in Table 1.[5]

ε	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
c_q	0.5	0.625	0.75	0.875	1	1.125	1.25	1.375	1.5
c_r	0.5286	0.6405	0.7567	0.8766	1	1.1263	1.2544	1.3835	1.5134
$c_{\bar{r}}$	0.5404	0.6473	0.7598	0.8774	1	1.1269	1.2565	1.3875	1.5195

Table 1: State of charge c under full information for different models: quantities, power rates with fixed trading periods, and power rates with flexible trading periods.

For $\varepsilon = -1$, the operator trades only $c = 1/2$ if relying on quantities and $c = 0.5286$ with flows. For $\varepsilon = 1$ the numbers are 3/2 and 1.5134, respectively. They are always higher under flows, give the storage operator more flexibility (and allow her to extract more surplus). This flexibility makes the most difference in extreme cases (up to 6%, or even 8% under flexible trading–see next Section): it either allows to charge higher prices and marginally increase quantities ($\varepsilon = 1$), or to better spread the price impact of sales ($\varepsilon = -1$).

3.4 Determining trading periods

So far we show that flows in continuous time provide flexibility in deciding when exactly to buy and sell in a (predetermined) trading window, to the benefit of the trader. This flexibility also extends to determining what *are* buying and selling periods. When time is discrete, a cycle is broken into a buying (sub-)period of low demand, and a selling (sub-)period of high demand; *duration* is irrelevant and so can never be adjusted. With continuous time, there is no such hardwired distinction. With full generality, the trader decides when to buy and sell as part of her optimal strategy. The buying and selling periods are no longer tied to the calendar.

[5] Flexible trading windows are analyzed in Section 3.4, forthcoming.

Instead, when to buy and sell are determined by optimality conditions–that is, exclusively by the relative difference between demand conditions and not by calendar time.[6] To be obvious, confining the supports of functions b to the interval $[0, \pi]$ and s to the interval $[\pi, 2\pi]$, as we do in Sections 3.1 to 3.3, is a substantive assumption that we now relax.

The point is best made by considering the same setting as in Lemma 5.[7] However Constraint (19) is relaxed to read instead

$$\int_0^T (b(t,\varepsilon) - s(t,\varepsilon))\, dt \geqslant 0 \qquad \text{for any } \; T \in [0, 2\pi] \; \text{ and } \; \varepsilon \in [-1, 1]. \tag{22}$$

Constraint (22) does not require the functions $b(t, \varepsilon)$ and $s(t, \varepsilon)$ to be everywhere positive, but only their difference to be non-negative over the interval $[0, T]$; that is, over that interval, the unit cannot sell more than it has purchased–there is no short selling. However, observing $b < 0$ before $t = \pi$ means that storage *sells before* $t = \pi$. On the other hand, a negative s after $t = \pi$ simply means that the operator *continues to buy* after $t = \pi$. We also note that $b(t, \varepsilon)$ must still remain positive close to $t = 0$ (no short-selling); this is guaranteed by the condition on the integral for small T.

Remark 6. *Why distinguish between $b > 0$ and $b < 0$ and likewise for the function $s(t, \varepsilon)$? The function s is cannot be defined before $t = \pi$ because there is no shock ε before π. We circumvent this problem by letting b and s be positive or negative.*

Before turning to the formal statement of the result, we provide an intuition of what may arise here. First, one expects continuous actions; in other words, $b(\pi, \varepsilon) = s(\pi, \varepsilon)$. This stands in contrast to the results of Section 3.3: in Lemma 5, for positive ε, buying abruptly stops at $t = \pi$ from $\frac{1}{2}(1 + \varepsilon) \sin t_s$ to zero and there is no trade until $t = \pi + t_s > \pi$. In the same manner, for any $\varepsilon < 0$, buying stops at $t = \pi - t_b < \pi$ and selling suddenly starts $t = \pi$ with a discrete jump $\frac{1}{2} \sin t_b$ at time t_b. Because we remove the restrictions to buy strictly up to $t = \pi$ and to sell strictly not before $t = \pi$, there is no need for sudden stops and jumps. The benefit of actions transitioning continuously from buying to selling, before or after π, is that the operator can better adjust her strategy. With this increased versatility, we expect the operator to trade more, compared to the case of Lemma 5.

[6]Except that one must precede the other.

[7]Presuming of full information is not as restrictive as may first appear; below we discuss how the results extend to incomplete information.

Lemma 7. *The optimal storage strategy that solves (18) subject to (22) is characterized as:*

- *for* $-1 \leqslant \varepsilon < 0$,

$$b(t,\varepsilon) = \begin{cases} \frac{1}{2}\left(\sin t - \sin \bar{t}_b\right) & \textit{if } \; t \in [\bar{t}_b, \pi], \\ 0 & \textit{otherwise,} \end{cases}$$

$$s(t,\varepsilon) = \begin{cases} \frac{1}{2}\left(-(1+\varepsilon)\sin t + \sin \bar{t}_b\right) & \textit{if } \; t \in [\pi, 2\pi], \\ 0 & \textit{otherwise;} \end{cases}$$

- *for* $0 \leqslant \varepsilon \leqslant 1$,

$$b(t,\varepsilon) = \begin{cases} \frac{1}{2}\left(\sin t + (1+\varepsilon)\sin \bar{t}_s\right) & \textit{if } \; t \in [0, \pi], \\ 0 & \textit{otherwise,} \end{cases}$$

$$s(t,\varepsilon) = \begin{cases} -\frac{1+\varepsilon}{2}\left(\sin t + \sin \bar{t}_s\right) & \textit{if } \; t \in [\pi, 2\pi - t_s], \\ 0 & \textit{otherwise;} \end{cases}$$

where $\bar{t}_b, \bar{t}_s \in [0, \pi/2]$ *are defined as roots of the equations*

$$\cos \bar{t}_b - (2\pi - \bar{t}_b)\sin \bar{t}_b = 1 + 2\varepsilon, \tag{23}$$

$$\cos \bar{t}_s - (2\pi - \bar{t}_s)\sin \bar{t}_s = \frac{1-\varepsilon}{1+\varepsilon}. \tag{24}$$

We see that the functions $b(t,\varepsilon), s(t,\varepsilon)$ are almost identical to those characterized in Lemma 5, up to the thresholds $\bar{t}_b, \bar{t}_s$. These are determined by different conditions (23), (24) than those of Lemma 5. That is, *when* exactly to buy and sell are the key differences.

From (23), if ε is negative, $\bar{t}_b > 0$; the operator does not want to buy a lot. However, $\bar{t}_b < t_b$: she starts buying *earlier* than in Lemma 5 (and stops buying later) because she still wants to purchase more than under the conditions of Lemma 5 (see Table 1). She can sell more, even under this same negative shock, because she can extend the selling window by starting to sell sooner–she is no longer restricted by the boundary at π. Indeed, the buying period ends at $t = \pi - \bar{t}_b$, where the function $b(t,\varepsilon)$ turns from positive to negative: the operator starts selling. Beyond $t = \pi$ the trader continues to sell under the function s until $t = 2\pi$. As for Lemma 5, the effects are strongest for more extreme shocks–see Table 1. Moreover, the selling function $s(t,\varepsilon)$ in Lemma 7 sits everywhere below its equivalent in Lemma 5: the benefit of selling earlier is to decrease the rate and so increase the price for all $t \in [\pi, 2\pi]$. Between higher prices and larger quantities, the trader generates more surplus. We show this case for the values of $\varepsilon = -9/10$ on the left-hand panel of Figure 3.

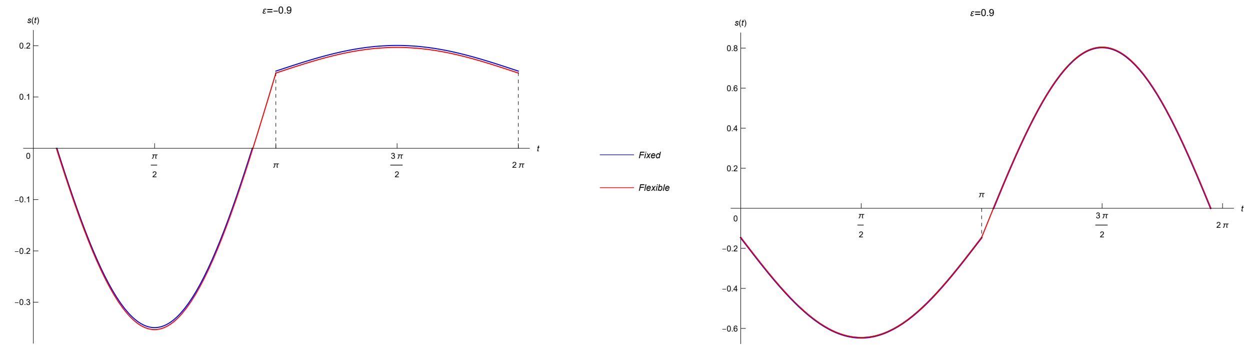

Figure 3: Buying and selling functions for fixed and flexible periods under $\varepsilon = -0.9$ and $\varepsilon = 0.9$.

Figure 3 compares the buying and selling functions for fixed (Lemma 5) and flexible (Lemma 7) trading periods under negative and positive shocks. For the negative shock ($\varepsilon = -9/10$), the operator starts selling slightly earlier under the flexible trading than under the fixed-window trading: $\bar{t}_b = 0.298 < 0.306 = t_b$. The buying process is shown by the functions $-b(t) < 0$ in red and blue, respectively. The flexible operator buys more than the fixed-window operator: the red curve sits everywhere below the blue one. The fixed-window operator stops at $\pi - t_b$ and remains idle until $t = \pi$. The flexible operator stops buying slightly later, at $\pi - \bar{t}_b$, and *immediately* starts selling. This is the same buying function $-b(t)$ but now it becomes positive, which denotes the selling mode (see Remark 6). While still selling, the flexible rate operator switches from function $-b(t)$ to function $s(t)$ at time $t = \pi$. The fixed-window rate operator also starts selling at $t = \pi$, and she sells slightly more at each particular moment of time $t \in [\pi, 2\pi]$, so the clearing price is slightly worse.

The case of positive shock ε is the mirror image: the operator starts buying as soon as possible–so, from $t = 0$–because she shows she can sell a large quantity later. Because she can buy for a longer duration, the buying rate $b(t, \varepsilon)$ is now lower than in Lemma 5 and the corresponding clearing price is lower. Indeed, the function $s(t, \varepsilon)$ in Lemma 7 is still negative starting from $t = \pi$, which means that storage continues to purchase beyond $t = \pi$. The sign of s changes at $t = \pi + \bar{t}_s$, where the operator switches from buying to selling. This switching point starts earlier than in Lemma 5 ($\bar{t}_s < t_s$): the trader can sell more. This also means that the selling function in Lemma 7 sits everywhere higher than the corresponding one in Lemma 5. Spreading the selling window to $[\pi + \bar{t}_s, 2\pi - \bar{t}_s]$ allows the trader to mitigate the price impact. Selling stops later, at $2\pi - \bar{t}_s > 2\pi - t_s$ and the unit remains idle until the end of the cycle.

On the right panel of Figure 3, both operators start to buy at $t = 0$ with a discrete jump. The fixed-window trader buys at a slightly higher rate throughout the entire purchasing period $[0, \pi]$. However, the flexible operator keeps buying (continuously) until $t = \pi + \bar{t}_s$, where $\bar{t}_s = 0.153$–and so buys a larger quantity. From that point, she switches to selling. The fixed-window operator finds it optimal to start to sell a little later: $t_s = 0.155 > \bar{t}_s$. In the interval $[\pi + t_s, 2\pi - t_s]$, the fixed-window operator sells at at a lower rate than the flexible operator, and so receives a higher price. But she misses out on selling between π and $\pi + t_s$.

We see that the benefits of using flows over quantities extend to correctly defining what is a trading window; this matters both for traded quantities and clearing prices. Importantly, this flexibility is *not* limited to the full-information case. Even if the details differ, it is easy to see that it also applies to the setting of incomplete information (Lemma 3), and even no information (Lemma 1) if the general demand function (1) is chosen to be asymmetric in expectation. For example, take $\mathbb{E}\varepsilon_2 = 0$; if $\varepsilon_1 < 0$, one can expect it is optimal to keep buying after $t = \pi$, and if $\varepsilon_1 > 0$, to start selling before $t = \pi$. Another form of asymmetric demand may be $\varepsilon_1 = 0$ but $\mathbb{E}\varepsilon_2 \neq 0$, to similar effects. We summarize our results.

Result. Modeling continuous flows is more flexible than discrete quantities and induces larger quantities to be traded. The differences between the quantity model and the flow model become sharper

- the more information about the shock the operator has;
- with the difference between the shock distributions and realizations for low and high demand.

Moreover, a model based on flows allows for the endogenous determination of trading periods.

4 Conclusion

This paper makes an important technical contribution to the question of modeling trade over time. An analyst can attack the same problem using a model of quantities (real numbers) or flows (functions of time) that lead to functional analysis.

Choosing one or the other is not completely self evident: the more demanding functional approach is better suited to a changing environment in that trading decisions better match demand conditions. Moreover, using flows rather than quantities allows the players to determine when to trade not based on time but on demand conditions; that is, trading periods are determined endogenously rather than exogenously by calendar time.

The choice of one approach over the other is not innocuous. Favoring flows and adopting a flexible trading approach both induce higher quantities to be traded. As ever, which to pick is left to the analyst; for long-horizon, computational problems, quantities are easier to handle and may be sufficient. On the other hand, we point that the presence of constraints that may bind at different times can be material and in such cases, flows may be better suited. While more demanding, flows also deliver the trader a higher surplus.

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Appendix–Proofs

Proof of Lemma 1. Since all the regularity constraints are satisfied, we can use the Kuhn-Tucker theorem. Note that $b(t) = 0$ for $\pi < t \leqslant 2\pi$ and $s(t) = 0$ for $0 \leqslant t < \pi$. Thus, we can set the same integration limits $[0, 2\pi]$ in both the objective functional and all the constraints, which allows us to solve the maximization problem within the integral. The Lagrangian is:

$$L = -\left(\theta - \sin t + b(t)\right) b(t) + \left(\theta - \sin t - s(t)\right) s(t) + \nu_1(t) b(t) + \nu_2(t) s(t) + \lambda \left(b(t) - s(t)\right),$$

with first-order conditions:

$$\begin{aligned} &-(\theta - \sin t) - 2b(t) + \nu_1(t) + \lambda = 0, \\ &\theta - \sin t - 2s(t) + \nu_2(t) - \lambda = 0, \end{aligned} \tag{25}$$

$$\int_0^{2\pi} \left(b(t) - s(t)\right) dt \geqslant 0. \tag{26}$$

We conjecture that there exist positive $t_b^1, t_b^2, t_s^1, t_s^2$, such that $0 \leqslant t_b^1 < \pi - t_b^2 \leqslant \pi + t_s^1 < 2\pi - t_s^2 \leqslant 2\pi$ and

$$\begin{cases} b(t) > 0 & \text{if } \ t \in (t_b^1; \pi - t_b^2), \\ b(t) = 0 & \text{otherwise}, \end{cases} \qquad \begin{cases} s(t) > 0 & \text{if } \ t \in (\pi + t_s^1; 2\pi - t_s^2), \\ s(t) = 0 & \text{otherwise}. \end{cases}$$

In the intervals $(t_b^1; \pi - t_b^2)$ and $(\pi + t_s^1; 2\pi - t_s^2)$, from (25) we obtain

$$b(t) = \frac{-\theta + \sin t + \lambda}{2}, \qquad s(t) = \frac{\theta - \sin t - \lambda}{2}. \tag{27}$$

By continuity, from (27) we have:

$$\begin{aligned} &-\theta + \sin t_b^1 + \lambda = -\theta + \sin t_b^2 + \lambda = 0, \\ &\theta + \sin t_s^1 - \lambda = \theta + \sin t_s^2 - \lambda = 0, \end{aligned}$$

which implies $t_b^1 = t_b^2$, $t_s^1 = t_s^2$, $\lambda = \theta - \sin t_b^1 = \theta + \sin t_s^1$, so $t_b^1 = t_s^1 = 0$ and

$$b(t) = \frac{\sin t}{2}, \qquad t \in (0, \pi); \qquad s(t) = -\frac{\sin t}{2}, \qquad t \in (\pi, 2\pi),$$

which concludes the proof. □

Proof of Lemma 2. Assume that we buy c units of energy, so $b = c$. Then for each c we need to maximize the function within the integral:

$$\max_{s(\varepsilon) \leqslant c} \left[\left(\pi\theta + 2(1+\varepsilon) - s(\varepsilon)\right) s(\varepsilon) \right]. \tag{28}$$

The solution to (28) is the following:

$$s_{max}(\varepsilon) = \begin{cases} \frac{\pi\theta}{2} + 1 + \varepsilon & \text{if} \quad \frac{\pi\theta}{2} + 1 + \varepsilon \leqslant c, \\ c & \text{if} \quad \frac{\pi\theta}{2} + 1 + \varepsilon > c, \end{cases}$$

where c is treated as a parameter. Thus, we have two different functions s_{max} for low (less than $c - 1 - \pi\theta/2$) and high ε. Three cases should be considered:

Case 1: $\boldsymbol{c - 1 - \frac{\pi\theta}{2} < -1}$ (or simply $c - \frac{\pi\theta}{2} < 0$), then $s_{max} = c$ for all ε. The final maximization problem turns into:

$$\max_c \left[-\left(\pi\theta - 2 + c\right) c + \frac{1}{2} \int_{-1}^{1} \left(\pi\theta + 2(1+\varepsilon) - c\right) c \, d\varepsilon \right],$$

which is equivalent to

$$\max_c \left[c(4 - 2c) \right],$$

and the solution is $c_{opt} = 1$.

Case 2: $\boldsymbol{-1 \leqslant c - 1 - \frac{\pi\theta}{2} \leqslant 1}$ (or $0 \leqslant c - \frac{\pi\theta}{2} \leqslant 2$), so we have two integrals: one with $s_{max} = c$ and another one with $s_{max} = \frac{\pi\theta}{2} + 1 + \varepsilon$. Then the final maximization problem becomes:

$$\max_c \Bigg[-\left(\pi\theta - 2 + c\right) c + \frac{1}{2} \int_{-1}^{c-1-\frac{\pi\theta}{2}} \left(\pi\theta + 2(1+\varepsilon) - \frac{\pi\theta}{2} - (1+\varepsilon)\right) \left(\frac{\pi\theta}{2} + 1 + \varepsilon\right) d\varepsilon \\ + \frac{1}{2} \int_{c-1-\frac{\pi\theta}{2}}^{1} \left(\pi\theta + 2(1+\varepsilon) - c\right) c \, d\varepsilon \Bigg],$$

which is equivalent to

$$\max_c \left[\frac{c^3}{6} - \left(2 + \frac{\pi\theta}{4}\right) c^2 + \left(4 + \frac{\pi^2\theta^2}{8}\right) c - \frac{\pi^3\theta^3}{48} \right].$$

The first order conditions give us $c_{max} = 4 + \frac{\pi\theta}{2} - 2\sqrt{2 + \pi\theta} < \frac{\pi\theta}{2}$, so the candidate solution for c_{max} lies outside of the required range. Hence, we have a corner solution here, the maximum value of which is lower than that of $c_{opt} = 1$.

Case 3: $\boldsymbol{c - 1 - \frac{\pi\theta}{2} > 1}$ (or $c - \frac{\pi\theta}{2} > 2$), so $s_{max} = \frac{\pi\theta}{2} + 1 + \varepsilon$ for all ε. Then the final maximization problem transforms into:

$$\max_c \left[-\left(\pi\theta - 2 + c\right) c + \frac{1}{2} \int_{-1}^{1} \left(\pi\theta + 2(1+\varepsilon) - \frac{\pi\theta}{2} - (1+\varepsilon)\right) \left(\frac{\pi\theta}{2} + 1 + \varepsilon\right) d\varepsilon \right],$$

which is equivalent to

$$\max_{c}\left[-c^2-(\pi\theta-2)c+\frac{4}{3}+\pi\theta+\frac{\pi^2\theta^2}{4}\right].$$

Within the range $c>2-\pi\theta/2$, this function monotonically decreases also, and there is no global optimal solution.

We conclude that $c_{opt}=1$ is the global maximum and the solution to this problem. $\square$

Proof of Lemma 3. Problem (12) may be reformulated as a three-step problem by introducing a new (interim) parameter $c\geqslant 0$, which denotes the current state of charge. This is the state variable of the system.

Step 1. We solve the problem

$$\min_{b_c(t)}\int_0^{\pi}\left(\theta-\sin t+b_c(t)\right)b_c(t)dt$$

subject to the constraints

$$b_c(t)\geqslant 0, \qquad \int_0^{\pi}b_c(t)dt=c$$

for each c.

Step 2. We find the function $s_c(t,\varepsilon)$ that solves the problem

$$\max_{s_c(t,\varepsilon)}\int_{\pi}^{2\pi}\left(\theta-(1+\varepsilon)\sin t-s_c(t,\varepsilon)\right)s_c(t,\varepsilon)dt$$

subject to the constraints

$$s_c(t,\varepsilon)\geqslant 0, \qquad \int_{\pi}^{2\pi}s_c(t,\varepsilon)dt\leqslant c$$

for each c.

Step 3. Finally, we find the optimal c that solves the problem

$$\max_{c\geqslant 0}\left[-\int_0^{\pi}\left(\theta-\sin t+b_c(t)\right)b_c(t)dt+\frac{1}{2}\int_{-1}^{1}\int_{\pi}^{2\pi}\left(\theta-(1+\varepsilon)\sin t-s_c(t,\varepsilon)\right)s_c(t,\varepsilon)dtd\varepsilon\right]. \tag{29}$$

Step 1 solution. In a fashion similar to the proof of Lemma 1, we construct the Lagrangian first (we omit subindices c throughout the entire proof):

$$L_b=\left(\theta-\sin t+b(t)\right)b(t)+\nu_1(t)b(t)+\mu_1\left(c/\pi-b(t)\right).$$

The first-order conditions are:

$$\theta - \sin t + 2b(t) + \nu_1(t) - \mu_1 = 0, \qquad \int_0^\pi b(t)dt = c. \tag{30}$$

Again, we conjecture that there exist t_b^1, t_b^2, such that $0 < t_b^1 < \pi - t_b^2 \leqslant \pi$ and

$$\begin{cases} b(t) > 0 & \text{if } \ t \in (t_b^1; \pi - t_b^2), \\ b(t) = 0 & \text{otherwise.} \end{cases}$$

In the interval $(t_b^1; \pi - t_b^2)$, from the first equation of (30) we obtain

$$b(t) = \frac{\mu_1 - \theta + \sin t}{2},$$

which implies, by continuity:

$$\mu_1 - \theta + \sin t_b^1 = \mu_1 - \theta + \sin t_b^2 = 0.$$

Thus, we have $t_b \equiv t_b^1 = t_b^2$ and $\mu_1 = \theta - \sin t_b$, so

$$b(t) = \frac{1}{2}\left(\sin t - \sin t_b\right), \qquad t \in [t_b, \pi - t_b].$$

From the second condition of (30), we can obtain the equation for finding t_b:

$$\int_0^\pi b(t)dt = c \quad \Rightarrow \quad \frac{1}{2}\int_{t_b}^{\pi - t_b} \left(\sin t - \sin t_b\right) dt = c \quad \Rightarrow \quad \cos t_b - \left(\frac{\pi}{2} - t_b\right)\sin t_b = c. \tag{31}$$

The last equality has a solution if and only if $c \leqslant 1$. For $c > 1$, the support of the function $b(t)$ extends up to $[0, \pi]$, and from (30) we have:

$$\frac{1}{2}\int_0^\pi \left(\mu_1 - \theta + \sin t\right) dt = c \quad \Rightarrow \quad \mu_1 = \frac{2}{\pi}(c-1) + \theta,$$

which implies

$$b(t) = \frac{\sin t}{2} + \frac{1}{\pi}(c-1), \qquad t \in [0, \pi].$$

Step 2 solution. Here, the Lagrangian is:

$$L_s = \left(\theta - (1+\varepsilon)\sin t - s(t,\varepsilon)\right) s(t,\varepsilon) + \nu_2(t)s(t,\varepsilon) + \mu_2\left(c/\pi - s(t,\varepsilon)\right).$$

The first-order conditions are:

$$\theta - (1+\varepsilon)\sin t - 2s(t,\varepsilon) + \nu_2(t) - \mu_2 = 0, \qquad \int_\pi^{2\pi} s(t,\varepsilon)dt \leqslant c. \tag{32}$$

For interior solutions, again, we conjecture that there exist t_s^1, t_s^2, such that $\pi < \pi + t_s^1 < 2\pi - t_s^2 \leqslant 2\pi$ and

$$\begin{cases} s(t,\varepsilon) > 0 & \text{if } \ t \in (\pi + t_s^1; 2\pi - t_s^2), \\ s(t,\varepsilon) = 0 & \text{otherwise.} \end{cases}$$

In the interval $(\pi + t_s^1; 2\pi - t_s^2)$, from the first equation of (32) we obtain

$$s(t,\varepsilon) = \frac{\theta - (1+\varepsilon)\sin t - \mu_2}{2},$$

which implies, by continuity:

$$\theta - (1+\varepsilon)\sin(\pi + t_s^1) - \mu_2 = \theta - (1+\varepsilon)\sin(2\pi - t_s^2) - \mu_2 = 0.$$

Thus, we have $t_s \equiv t_s^1 = t_s^2$ and (since $\sin(2\pi - t_s) = -\sin t_s$) we have $\mu_2 = \theta + (1+\varepsilon)\sin t_s$, so

$$s(t,\varepsilon) = -\frac{1+\varepsilon}{2}\left(\sin t + \sin t_s\right), \qquad t \in (\pi + t_s, 2\pi - t_s).$$

Since $\mu_2 > 0$ for any values of the parameters, the second condition of (32) binds, and from it we can obtain the equation that implicitly defines t_s:

$$\int_\pi^{2\pi} s(t,\varepsilon)dt = c \ \Rightarrow\ -\frac{1+\varepsilon}{2}\int_{\pi+t_s}^{2\pi-t_s}\left(\sin t + \sin t_s\right)dt = c \ \Rightarrow\ \cos t_s - \left(\frac{\pi}{2} - t_s\right)\sin t_s = \frac{c}{1+\varepsilon}. \tag{33}$$

The last equality has a solution if and only if $\varepsilon \geqslant c - 1$. When $\varepsilon < c - 1$, one faces a corner solution.

For corner solutions, the support of the function $s(t,\varepsilon)$ extends up to $[0,\pi]$, and from (32) we have:

$$\frac{1}{2}\int_\pi^{2\pi}\left(\theta - (1+\varepsilon)\sin t - \mu_2\right)dt \leqslant c \quad \Rightarrow \quad \frac{\pi}{2}(\theta - \mu_2) + 1 + \varepsilon \leqslant c.$$

When $\mu_2 > 0$, this condition binds and we have

$$\mu_2 = \theta - \frac{2}{\pi}\left(c - 1 - \varepsilon\right) \quad \Rightarrow \quad s(t,\varepsilon) = \frac{1}{\pi}(c - 1 - \varepsilon) - \frac{1+\varepsilon}{2}\sin t,$$

which holds for $c - 1 - \pi\theta/2 \leqslant \varepsilon < c - 1$. Whenever $1 + \varepsilon + \pi\theta/2 < c$, $\mu_2 = 0$ and we have

$$s(t,\varepsilon) = \frac{\theta - (1+\varepsilon)\sin t}{2},$$

which requires $\varepsilon < c-1-\pi\theta/2$. Next we turn to the determination of the optimal quantity c.

Step 3 solution. Let's first summarize steps 1 and 2. We obtained solutions for $b(t)$ that differ depending on $c \leqslant 1$ and $c > 1$; let's denote them as $b_1(t)$ and $b_2(t)$, respectively. We also obtained solutions for $s(t,\varepsilon)$ that also differ depending on whether $\varepsilon \geqslant c-1$, $c-1-\pi\theta/2 \leqslant \varepsilon < c-1$, and $\varepsilon < c-1-\pi\theta/2$ that we correspondingly denote as $s_1(t,\varepsilon)$, $s_2(t,\varepsilon)$, and $s_3(t,\varepsilon)$. Hence we end with six cases to consider and find the value of c that maximizes the equation in (29) for each of them:

- $c \leqslant 1$;
- $1 < c \leqslant \min\{2, \frac{\pi\theta}{2}\}$;
- $2 < c \leqslant \frac{\pi\theta}{2}$ (if $2 < \frac{\pi\theta}{2}$);
- $\frac{\pi\theta}{2} < c \leqslant 2$ (if $\frac{\pi\theta}{2} < 2$);
- $\max\{2, \frac{\pi\theta}{2}\} < c \leqslant 2 + \frac{\pi\theta}{2}$;
- $c > 2 + \frac{\pi\theta}{2}$.

$\boldsymbol{c \leqslant 1}$. We need to find c that solves the following:

$$\max_{c\in[0,1]} \left[-\int_0^{\pi} (\theta - \sin t + b_1(t))\, b_1(t) dt + \frac{1}{2}\int_{c-1}^{1}\int_{\pi}^{2\pi} (\theta - (1+\varepsilon)\sin t - s_1(t,\varepsilon))\, s_1(t,\varepsilon) dt d\varepsilon + \right.$$
$$\left. + \frac{1}{2}\int_{-1}^{c-1}\int_{\pi}^{2\pi} (\theta - (1+\varepsilon)\sin t - s_2(t,\varepsilon))\, s_2(t,\varepsilon) dt d\varepsilon \right].$$

Let's denote the three summands here as B_1, S_1, and $\overline{S}_2$, respectively. We have

$$\begin{aligned}
B_1 &= -\frac{1}{2}\int_{t_b}^{\pi-t_b} \left(\theta - \sin t + \frac{1}{2}(\sin t - \sin t_b)\right)(\sin t - \sin t_b)\, dt \\
&= -\frac{1}{4}\left((4\theta - \sin t_b)\cos t_b - \left(\frac{\pi}{2} - t_b\right)(\cos 2t_b + 4\theta \sin t_b)\right), \\
S_1 &= -\frac{1}{2}\int_{c-1}^{1} \frac{1+\varepsilon}{2}\int_{\pi+t_s}^{2\pi-t_s} \left(\theta - (1+\varepsilon)\sin t + \frac{1+\varepsilon}{2}(\sin t + \sin t_s)\right)(\sin t + \sin t_s)\, dt d\varepsilon \\
&= \frac{1}{8}\int_{c-1}^{1} (1+\varepsilon)\left((4\theta + (1+\varepsilon)\sin t_s)\cos t_s + \left(\frac{\pi}{2} - t_s\right)((1+\varepsilon)\cos 2t_s - 4\theta\sin t_s)\right) d\varepsilon, \\
\overline{S}_2 &= \frac{1}{2}\int_{-1}^{c-1}\int_{\pi}^{2\pi} \left(\theta - (1+\varepsilon)\sin t - \frac{1}{\pi}(c-1-\varepsilon) + \frac{1+\varepsilon}{2}\sin t\right)\left(\frac{1}{\pi}(c-1-\varepsilon) - \frac{1+\varepsilon}{2}\sin t\right) d\varepsilon \\
&= \frac{c^2}{2}\left(\theta + \frac{c}{3}\left(\frac{\pi}{8} - \frac{1}{\pi}\right)\right).
\end{aligned}$$

To differentiate with respect to c, we need to find the derivatives $(t_b)'_c$ and $(t_s)'_c$ of the thresholds t_b, t_s, which are implicitly defined. From (31) and (33), we obtain

$$(t_b)'_c = -\frac{1}{\left(\frac{\pi}{2} - t_b\right)\cos t_b}, \qquad (t_s)'_c = -\frac{1}{(1+\varepsilon)\left(\frac{\pi}{2} - t_s\right)\cos t_s}.$$

Then, using the Leibniz integral rule for S_1,

$$\begin{aligned}
(B_1)'_c &= -\frac{1}{\left(\frac{\pi}{2} - t_b\right)\cos t_b} \cdot \left(\frac{\pi}{2} - t_b\right)\cos t_b\,(\theta - \sin t_b) = -\theta + \sin t_b, \\
(S_1)'_c &= -\frac{c}{2}\left(\theta + \frac{\pi c}{8}\right) + \frac{1}{2}\int_{c-1}^{1} \frac{1}{(1+\varepsilon)\left(\frac{\pi}{2} - t_s\right)\cos t_s} \cdot \left(\frac{\pi}{2} - t_s\right)(1+\varepsilon)\cos t_s\,(\theta + (1+\varepsilon)\sin t_s)\,d\varepsilon \\
&= \theta(1-c) - \frac{\pi c^2}{16} + \frac{1}{2}\int_{c-1}^{1}(1+\varepsilon)\sin t_s d\varepsilon, \\
\left(\overline{S}_2\right)'_c &= \theta c + \frac{c^2}{16\pi}\left(\pi^2 - 8\right).
\end{aligned}$$

Thus, the first derivative of the entire maximand is

$$\left(B_1 + S_1 + \overline{S}_2\right)'_c = \sin t_b + \frac{1}{2}\int_{c-1}^{1}(1+\varepsilon)\sin t_s d\varepsilon - \frac{c^2}{2\pi}.$$

We can see that $\sin t_b$ is always nonnegative and it decreases monotonically with increasing c. The second summand is also nonnegative, and its derivative with respect to c is equal to

$$\left(\frac{1}{2}\int_{c-1}^{1}(1+\varepsilon)\sin t_s d\varepsilon - \frac{c^2}{2\pi}\right)'_c = -\frac{1}{2}\int_{c-1}^{1}\frac{d\varepsilon}{\frac{\pi}{2} - t_s} < 0,$$

so this integral also decreases monotonically with increasing c. Thus, the entire first derivative decreases with c and reaches its minimum value at the boundary $c = 1$. This value may be computed:

$$\sin\ 0 + \frac{1}{2}\int_{0}^{1}(1+\varepsilon)\sin t_s(1) d\varepsilon - \frac{1}{2\pi} \approx \frac{1}{2}\cdot 0.349 - 0.159 = 0.015,$$

where $t_s(1)$ is the root of the equation $\cos t_s - \left(\frac{\pi}{2} - t_s\right)\sin t_s = \frac{1}{1+\varepsilon}$. We can see that for any c the first derivative of the maximand is positive and, thus, the latter reaches its maximum at $c = 1$.

$\mathbf{1 < c \leqslant \min\{2, \frac{\pi\theta}{2}\}}$. We need to find c that solves the following:

$$\max_{c \in \left[1, \min\{2, \frac{\pi\theta}{2}\}\right]} \left[-\int_0^{\pi}(\theta - \sin t + b_2(t))\,b_2(t)dt + S_1 + \overline{S}_2\right].$$

Let's denote the first integral of the maximand as B_2. We have

$$B_2 = -\int_0^{\pi}\left(\theta - \sin t + \frac{\sin t}{2} + \frac{1}{\pi}(c-1)\right)\left(\frac{\sin t}{2} + \frac{1}{\pi}(c-1)\right)dt = \frac{\pi}{8} - \theta c - \frac{(c-1)^2}{\pi},$$

and

$$(B_2)'_c = -\theta - \frac{2}{\pi}(c-1).$$

The first derivative of the entire maximand is

$$\left(B_2 + S_1 + \overline{S}_2\right)'_c = \frac{1}{2}\int_{c-1}^{1} (1+\varepsilon)\sin t_s d\varepsilon - \frac{c^2 + 4(c-1)}{2\pi}.$$

Equalizing this to zero, we get the formula (14). Let the root of this equation be c_m. We can compute it: $c_m \approx 1.009$. The first derivative of the maximand is positive if $c < c_m$ and negative if $c > c_m$. Thus, c_m is the optimal state of charge for the given interval.

$\boldsymbol{2 < c \leqslant \frac{\pi\theta}{2}}$ (if $2 < \frac{\pi\theta}{2}$). We need to find c that solves the following:

$$\max_{c \in \left[2, \frac{\pi\theta}{2}\right]} \left[B_2 + \frac{1}{2}\int_{-1}^{1}\int_{\pi}^{2\pi} \left(\theta - (1+\varepsilon)\sin t - s_2(t,\varepsilon)\right) s_2(t,\varepsilon) d\varepsilon \right].$$

Let's denote the last summand as S_2. We have

$$\begin{aligned} S_2 &= \frac{1}{2}\int_{-1}^{1}\int_{\pi}^{2\pi} \left(\theta - (1+\varepsilon)\sin t - \frac{1}{\pi}(c-1-\varepsilon) + \frac{1+\varepsilon}{2}\sin t\right)\left(\frac{1}{\pi}(c-1-\varepsilon) - \frac{1+\varepsilon}{2}\sin t\right) d\varepsilon \\ &= \frac{\pi^2 - 8}{6\pi} + c\left(\theta + \frac{2-c}{\pi}\right). \end{aligned}$$

Then

$$(B_2 + S_2)'_c = -\frac{4(c-1)}{\pi} < 0.$$

$\boldsymbol{\frac{\pi\theta}{2} < c \leqslant 2}$ (if $\frac{\pi\theta}{2} < 2$). We need to find c that solves the following:

$$\begin{aligned} \max_{c \in \left[\frac{\pi\theta}{2}, 2\right]} \quad & \left[B_2 + S_1 + \frac{1}{2}\int_{c-1-\frac{\pi\theta}{2}}^{c-1}\int_{\pi}^{2\pi} \left(\theta - (1+\varepsilon)\sin t - s_2(t,\varepsilon)\right) s_2(t,\varepsilon) d\varepsilon \right. \\ & \left. + \frac{1}{2}\int_{-1}^{c-1-\frac{\pi\theta}{2}}\int_{\pi}^{2\pi} \left(\theta - (1+\varepsilon)\sin t - s_3(t,\varepsilon)\right) s_3(t,\varepsilon) d\varepsilon \right]. \end{aligned}$$

Let the last two summands of the maximized function be $\underline{S}_2$ and $\overline{S}_3$, respectively. We have

$$\begin{aligned} \underline{S}_2 &= \frac{1}{2}\int_{c-1-\frac{\pi\theta}{2}}^{c-1}\int_{\pi}^{2\pi} \left(\theta - (1+\varepsilon)\sin t - \frac{1}{\pi}(c-1-\varepsilon) + \frac{1+\varepsilon}{2}\sin t\right)\left(\frac{1}{\pi}(c-1-\varepsilon) - \frac{1+\varepsilon}{2}\sin t\right) d\varepsilon \\ &= \frac{\pi^2\theta^3}{384}\left(\pi^2 - 8\right) + \frac{\pi\theta}{32}c\left(8\theta + \pi\left(c - \frac{\pi\theta}{2}\right)\right), \\ \overline{S}_3 &= \frac{1}{2}\int_{-1}^{c-1-\frac{\pi\theta}{2}}\int_{\pi}^{2\pi} \left(\theta - (1+\varepsilon)\sin t - \frac{\theta - (1+\varepsilon)\sin t}{2}\right)\frac{\theta - (1+\varepsilon)\sin t}{2} d\varepsilon \\ &= \frac{1}{4}\left(c - \frac{\pi\theta}{2}\right)\left(\theta c + \frac{\pi}{12}\left(c - \frac{\pi\theta}{2}\right)^2\right). \end{aligned}$$

Then

$$\begin{aligned}\left(B_2+S_1+\overline{S}_2+\overline{S}_3\right)'_c &= -\theta-\frac{2}{\pi}(c-1)+\theta(1-c)-\frac{\pi c^2}{16}+\frac{1}{2}\int_{c-1}^{1}(1+\varepsilon)\sin t_s d\varepsilon\\ &+ \frac{\pi\theta}{64}\left(4\pi c+(16-\pi^2)\theta\right)+\frac{1}{64}\left(4\pi c^2+(\pi^2-8)(\pi\theta-4c)\theta\right)\\ &= \frac{1}{2}\int_{c-1}^{1}(1+\varepsilon)\sin t_s d\varepsilon-\left(\frac{\pi\theta^2}{8}+\frac{2(c-1)}{\pi}+\frac{\theta}{2}\left(c-\frac{\pi\theta}{2}\right)\right).\end{aligned}$$

Even if we take 1 as an upper bound of $\sin t_s$ and put the minimum possible $c=\pi\theta/2$ to maximize the integral, the resulting estimation

$$\frac{1}{2}\int_{\frac{\pi\theta}{2}-1}^{1}(1+\varepsilon)d\varepsilon=1-\frac{\pi^2\theta^2}{16}$$

will be lower than even the first summand $\pi\theta^2/8$ of the second part of the derivative for any $\theta>1$. Thus, the maximand always decreases by c in the interval $[\pi\theta/2,2]$.

$\mathbf{\max\{2,\frac{\pi\theta}{2}\}<c\leqslant 2+\frac{\pi\theta}{2}}$. We need to find c that solves the following:

$$\max_{c\in\left[\max\{2,\frac{\pi\theta}{2}\},2+\frac{\pi\theta}{2}\right]}\left[B_2+\frac{1}{2}\int_{c-1-\frac{\pi\theta}{2}}^{1}\int_{\pi}^{2\pi}\left(\theta-(1+\varepsilon)\sin t-s_2(t,\varepsilon)\right)s_2(t,\varepsilon)dtd\varepsilon+\overline{S_3}\right].$$

Let's denote the middle summand as $\underline{S}_2$. We have

$$\begin{aligned}\underline{S}_2 &= \frac{1}{2}\int_{c-1-\frac{\pi\theta}{2}}^{1}\int_{\pi}^{2\pi}\left(\theta-(1+\varepsilon)\sin t-\frac{1}{\pi}(c-1-\varepsilon)+\frac{1+\varepsilon}{2}\sin t\right)\left(\frac{1}{\pi}(c-1-\varepsilon)-\frac{1+\varepsilon}{2}\sin t\right)d\varepsilon\\ &= \frac{2+\frac{\pi\theta}{2}-c}{16\pi}\left(8\left(2+\frac{\pi\theta}{2}\right)c+\frac{\pi^2-8}{3}\left(4+2\left(c-\frac{\pi\theta}{2}\right)+\left(c-\frac{\pi\theta}{2}\right)^2\right)\right).\end{aligned}$$

Then

$$\left(B_2+\underline{S}_2+\overline{S}_3\right)'_c=\frac{1}{2\pi}\left(\left(c-\frac{\pi\theta}{2}\right)^2-8(c-1)\right).$$

Since inside the given interval $\left(c-\frac{\pi\theta}{2}\right)^2<4$ and $8(c-1)>8$, this derivative is also negative and the maximand decreases for the given c.

$\mathbf{c>2+\frac{\pi\theta}{2}}$. We need to find c that solves the following:

$$\max_{c>2+\frac{\pi\theta}{2}}\left[B_2+\frac{1}{2}\int_{-1}^{1}\int_{\pi}^{2\pi}\left(\theta-(1+\varepsilon)\sin t-s_3(t,\varepsilon)\right)s_3(t,\varepsilon)d\varepsilon\right].$$

Let's denote the last summand as S_3. We have

$$S_3=\frac{1}{2}\int_{-1}^{1}\int_{\pi}^{2\pi}\left(\theta-(1+\varepsilon)\sin t-\frac{\theta-(1+\varepsilon)\sin t}{2}\right)\frac{\theta-(1+\varepsilon)\sin t}{2}d\varepsilon=\frac{\pi}{6}+\theta+\frac{\pi\theta^2}{4}.$$

Then

$$(B_2 + S_3)'_c = -\theta - \frac{2}{\pi}(c-1) < 0.$$

Since the maximized function is continuous for any c (the values are always equal at the edges of the adjacent intervals), we may conclude that it monotonically increases for any c up to c_m and monotonically decreases afterwards. Thus, c_m is a global maximum. If $k \leqslant c_m$, the storage unit always buys (and sells) up to its capacity. However, in case $k > c_m$ the optimal strategy of storage is not to purchase up to the entire capacity k but to buy (and sell) only c_m units of energy. The optimal strategies are $b_2(t)$ and either $s_1(t, \varepsilon)$ or $s_2(t, \varepsilon)$ depending on ε. □

Proof of Lemma 4. For any ε we must have $S(\varepsilon) = B(\varepsilon)$: for whatever reason, if not planning to sell everything, then storage operator need not buy that much. Then the maximization problem (15) simplifies to:

$$\max_{B(\varepsilon)} \left[(4 + 2\varepsilon)\, B(\varepsilon) - 2B^2(\varepsilon)\right],$$

with the obvious solution (17). □

Proof of Lemma 5. Since all the regularity constraints are satisfied, we can use the Kuhn-Tucker theorem here. Note that $b(t) = 0$ for $\pi < t \leqslant 2\pi$ and $s(t) = 0$ for $0 \leqslant t < \pi$. Thus, we can set the same integration limits $[0, 2\pi]$ in both the objective functional and all the constraints, which allows us to solve the maximization problem within the integral. For any $\varepsilon \in [-1, 1]$, the Lagrangian is:

$$L = -\left(\theta - \sin t + b(t,\varepsilon)\right) b(t,\varepsilon) \\ + \left(\theta - \sin t - s(t,\varepsilon)\right) s(t,\varepsilon) + \nu_1(t) b(t,\varepsilon) + \nu_2(t) s(t,\varepsilon) + \lambda \left(b(t,\varepsilon) - s(t,\varepsilon)\right).$$

The first-order conditions with respect to b and s are as follows:

$$\begin{aligned} &-(\theta - \sin t) - 2b + \nu_1(t) + \lambda = 0, \\ &\theta - (1+\varepsilon)\sin t - 2s + \nu_2(t) - \lambda = 0, \end{aligned} \tag{34}$$

$$\int_0^{2\pi} \left(b(t) - s(t)\right) dt = 0. \tag{35}$$

Inequality (35) binds due to full information: if the operator knows that she is not going to sell everything, she would simply buy less. We consider four different cases:

1. Restricted support for both b and s: there exist positive t_b and t_s, such that $0 < t_b < \pi - t_b < \pi + t_s < 2\pi - t_s < 2\pi$ (the symmetry of support is provided by the symmetry of sin function; see the proof of Lemma 1);
2. Restricted support only for b and full support $[0, \pi]$ for s;
3. Full support $[0, \pi]$ for b and restricted support for s;
4. Full support for both b and s.

1. From (34), we have $\lambda = \theta - \sin t_b$, $\lambda = \theta + (1 + \varepsilon) \sin t_s$, which implies $\sin t_b = -(1 + \varepsilon) \sin t_s$. The last equality is impossible for $t_b, t_s > 0$.

2. From the first equation of (34), we have $\lambda = \theta - \sin t_b$, which implies the following form of functions b and s:

$$b(t, \varepsilon) = \frac{1}{2} \left(\sin t - \sin t_b \right), \qquad t \in [t_b, \pi - t_b],$$
$$s(t, \varepsilon) = -\frac{1}{2} \left((1 + \varepsilon) \sin t + \sin t_b \right), \qquad t \in [\pi, 2\pi].$$

From (35), we can find t_b:

$$\int_0^{2\pi} (b(t) - s(t))\, dt = \cos t_b - (\pi - t_b) \sin t_b - 1 - \varepsilon = 0,$$

which is equivalent to (20) and has solutions only if $-1 \leqslant \varepsilon \leqslant 0$.

3. From the second equation of (34), we have $\lambda = \theta + (1 + \varepsilon) \sin t_s$, which implies the following form of functions b and s:

$$b(t, \varepsilon) = \frac{1}{2} \left(\sin t + (1 + \varepsilon) \sin t_s \right), \qquad t \in [0, \pi],$$
$$s(t, \varepsilon) = -\frac{1 + \varepsilon}{2} \left(\sin t + \sin t_s \right) \qquad t \in [\pi + t_s, 2\pi - t_s].$$

From (35), we can find t_s:

$$\int_0^{2\pi} (b(t) - s(t))\, dt = 1 - (1 + \varepsilon) \cos t_s - (1 + \varepsilon)(\pi - t_s) \sin t_s = 0,$$

which is equivalent to (21) and has solutions only if $0 \leqslant \varepsilon \leqslant 1$.

4. From (34), we have

$$b(t, \varepsilon) = \frac{1}{2} \left(\lambda - \theta + \sin t \right), \qquad t \in [0, \pi],$$
$$s(t, \varepsilon) = -\frac{1}{2} \left(\lambda - \theta + (1 + \varepsilon) \sin t \right) \qquad t \in [\pi, 2\pi].$$

From (35), we can find λ:

$$\int_0^{2\pi} (b(t) - s(t))\, dt = \pi\lambda - \pi\theta - \varepsilon = 0,$$

which implies $\lambda = \varepsilon/\pi + \theta$. However, in this case

$$s(t, \varepsilon) = -\frac{1}{2}\left(\frac{\varepsilon}{\pi} + (1 + \varepsilon)\sin t\right) < 0,$$

which makes this case invalid. □

Proof of Lemma 7. The proof almost repeats that of Lemma 5 with one important distinction. Since the function $b(t, \varepsilon)$ now turns smoothly into the function $s(t, \varepsilon)$, there are no boundaries near $t = \pi$. Thus, we should consider only two cases: when we start buying later from some $\bar{t}_b$ and when we stop selling earlier at some $2\pi - \bar{t}_s$: $0 \leqslant \bar{t}_b, \bar{t}_s < \pi/2$. The first-order conditions and the constraint on the capacity level are the same as (34) and (35).

1. $\bar{t}_b$ case. From the first equation of (34), we have $\lambda = \theta - \sin \bar{t}_b$, which implies the following form of functions b and s:

$$b(t, \varepsilon) = \frac{1}{2}\left(\sin t - \sin \bar{t}_b\right), \qquad t \in [\bar{t}_b, \pi],$$
$$s(t, \varepsilon) = -\frac{1}{2}\left((1 + \varepsilon)\sin t + \sin \bar{t}_b\right), \qquad t \in [\pi, 2\pi].$$

From (35), we can find $\bar{t}_b$:

$$\int_0^{2\pi} (b(t) - s(t))\, dt = \frac{1}{2}\big(\cos \bar{t}_b - (2\pi - \bar{t}_b)\sin \bar{t}_b - 1 - 2\varepsilon\big) = 0,$$

which is equivalent to (23) and has solutions only if $-1 \leqslant \varepsilon \leqslant 0$.

2. $\bar{t}_s$ case. From the second equation of (34), we have $\lambda = \theta + (1 + \varepsilon)\sin \bar{t}_s$, which implies the following form of functions b and s:

$$b(t, \varepsilon) = \frac{1}{2}\left(\sin t + (1 + \varepsilon)\sin \bar{t}_s\right), \qquad t \in [0, \pi],$$
$$s(t, \varepsilon) = -\frac{1 + \varepsilon}{2}\left(\sin t + \sin \bar{t}_s\right) \qquad t \in [\pi, 2\pi - \bar{t}_s].$$

From (35), we can find $\bar{t}_s$:

$$\int_0^{2\pi} (b(t) - s(t))\, dt = \frac{1}{2}\big(1 - \varepsilon - (1 + \varepsilon)\cos \bar{t}_s + (1 + \varepsilon)(2\pi - \bar{t}_s)\sin \bar{t}_s\big) = 0,$$

which is equivalent to (24) and has solutions only if $0 \leqslant \varepsilon \leqslant 1$. □